CS 170 Discussion 1 (Fall 2017)

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Asymptotic Analysis

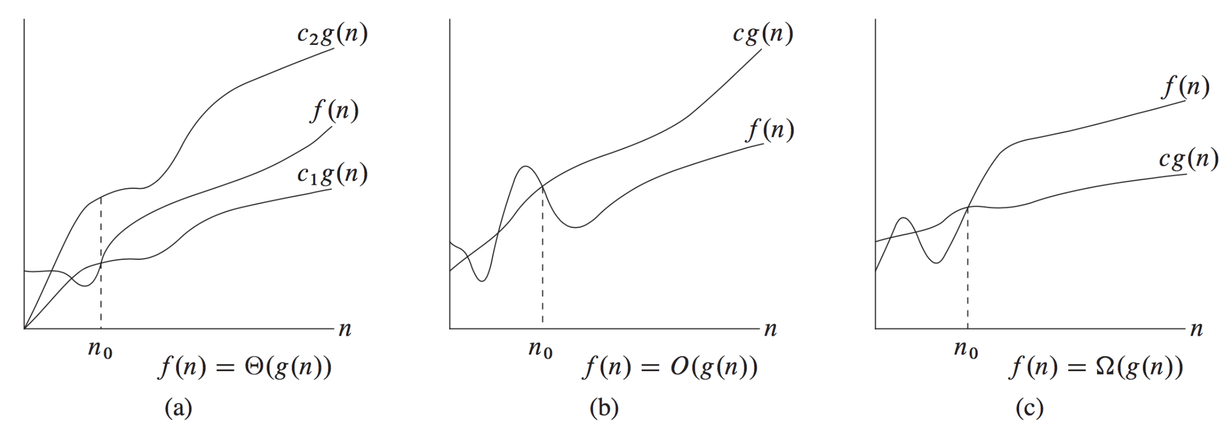
When looking at function behavior, we want to thing about how it behaves when the input gets significantly large. We use the following notations O ("Big-Oh"), Ω ("Big-Omega"), and Θ ("Big-Theta"). These refer to function sets rather than runtime specifically.

- $O(\cdot)$
  This is considered an "upper bound". $f(n) \in O(g(n))$ means that the function $f(n)$ belongs in the set of functions that are upper bounded by $g(n)$ when n gets significantly large.
  Mathematically, the can be referred to as $f(n) \leq c \cdot g(n)$ for all $n \geq n_0$ for some constant c and some n_0.
  If $f(n) \in O(n^2)$, it also means that $f(n) \in O(n^3)$, $f(n) \in O(2^n)$, and $f(n) \in O(\cdot)$ of any function that upper bounds n^2.

- $\Omega(\cdot)$
  This is a "lower bound". $f(n) \in \Omega(g(n))$ means that $f(n)$ is lower bounded by $g(n)$. $f(n) \geq c \cdot g(n)$ for all $n \geq n_0$for some constant c and some n_0. Like $O(\cdot)$, if $f(n) \in \Omega(n^2)$, then $f(n) \in \Omega(n)$, $f(n) \in \Omega(\log n)$, and any function that lower bounds n^2.

- $\Theta(\cdot)$
  This is a "tight bound". $f(n) \in \Theta(g(n)) \Leftrightarrow f(n) \in \Omega(g(n))$ and $f(n) \in O(g(n))$. This mathematical expression is $c_1 \cdot g(n) \leq f(n) \leq c_2 \cdot g(n)$for all $n \geq n_0$ for some n_0 where c_1 and c_2 are constants and $c_1 \leq c_2$.

It is important to note that the bounds are for when the input size grows significantly large. For example $f = 1000n^2$ and $g = n^3$, at smaller values of n, $f(n)$ dominates $g(n)$. But past a certain n_0, $g(n)$ will always upper bound $f(n)$. Below are some more examples.



(from Introduction to Algorithms by Cormen, Leiserson, Rivest, Stein)

Here are some rules when dealing with asymptotic analysis

- Remove multiplicative constants and lower order terms.
  e.g. $O(2n^4 + n^2 + n \log n) = O(n^4)$.

- Any exponential dominates any polynomial.
  e.g. $n^2 \in O(2^n)$.

- Any polynomial dominates any logarithm.
  e.g. $n^2 \in \Omega(\log_3 n)$.

- Removing exponents.
  e.g. $x^b = 2^{\log_2(x^b)} = 2^{b \log x}$

- Using limits and ratios.
  We can compare the ratio of $f(n)$ over $g(n)$.

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = \begin{cases} 0, & & f(n) \in O(g(n)) \\ c, & 0 < c < \infty & f(n) \in \Theta(g(n)) \\ \infty, & & f(n) \in \Omega(g(n)) \end{cases}$$

  e.g.

$$\begin{aligned} f(n) &= n \\ g(n) &= \log n \\ \lim_{n\to\infty} \frac{f(n)}{g(n)} &= \frac{n}{\log n} \\ &\to \frac{1}{n^{-1}} = n \to \infty \\ \therefore f(n) &\in \Omega(g(n)) \end{aligned}$$