CS 170 DISCUSSION 9

LINEAR PROGRAM AND DUALITY

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LINEAR PROGRAMMING

- Decision Variables
  - Things you have control over.
- Linear Objective Function
  - Optimize with respect to decision variables. Maximize or minimize.
- Constraints
  - Inequality or Equality
  - Restrictions for decision variables.

LINEAR PROGRAMMING

- General form of linear program.
- Turn ≥ into ≤ by multiplying by -1
  - $g(x) \geq b$ —> $-g(x) \leq -b$
- Turn an equality into two inequalities
  - $g(x) = b$
    - $g(x) \leq b$ and $g(x) \geq b$
    - —> $g(x) \leq b$ and $-g(x) \leq -b$
- Can also move b to LHS.

$$\begin{gathered}\max x_1 + 6x_2 \\ \text{s.t. } x_1 \leq 200 \\ x_2 \leq 300 \\ x_1 + x_2 \leq 400 \\ x_1, x_2 \geq 0\end{gathered}$$

$$\begin{gathered}\max x_1 + 6_2 \\ \text{s.t. } x_1 - 200 \leq 0 \\ x_2 - 300 \leq 0 \\ x_1 + x_2 - 400 \leq 0 \\ x_1, x_2 \geq 0\end{gathered}$$

LINEAR PROGRAMMING

- Other variants:
- Going between max and min by multiplying by -1 to objective function.
- Inequalities can turned into equalities by introducing slack variable.
  - $x_1 + x_2 \le b$
    - $x_1 + x_2 + s = b$ and $s \ge 0$
- Decision variable x has no sign restrictions.
  - Add x^+ and x^- that corresponds to positive and negative of decision variable x.
  - $x^+, x^- \ge 0$
  - Replace x with $x^+ - x^-$ in LP.

$$\max x_1 + 6x_2$$
$$\text{s.t. } x_1 \le 200$$
$$x_2 \le 300$$
$$x_1 + x_2 \le 400$$
$$x_1, x_2 \ge 0$$

$$\max x_1 + 6_2$$
$$\text{s.t. } x_1 - 200 \le 0$$
$$x_2 - 300 \le 0$$
$$x_1 + x_2 - 400 \le 0$$
$$x_1, x_2 \ge 0$$

LINEAR PROGRAMMING

$$\begin{aligned} &\max\ x_1 + 6x_2 \\ &\text{s.t.} \\ &x_1 \le 200 \\ &x_2 \le 300 \\ &x_1 + x_2 \le 400 \\ &x_1, x_2 \ge 0 \end{aligned}$$

$$\begin{aligned} &\min\ -x_1 - 6x_2 \\ &\text{s.t.} \\ &x_1 + s_1 = 200 \\ &x_2 + s_2 = 300 \\ &x_1 + x_2 + s_3 = 400 \\ &x_1, x_2, s_1, s_2, s_3 \ge 0 \end{aligned}$$

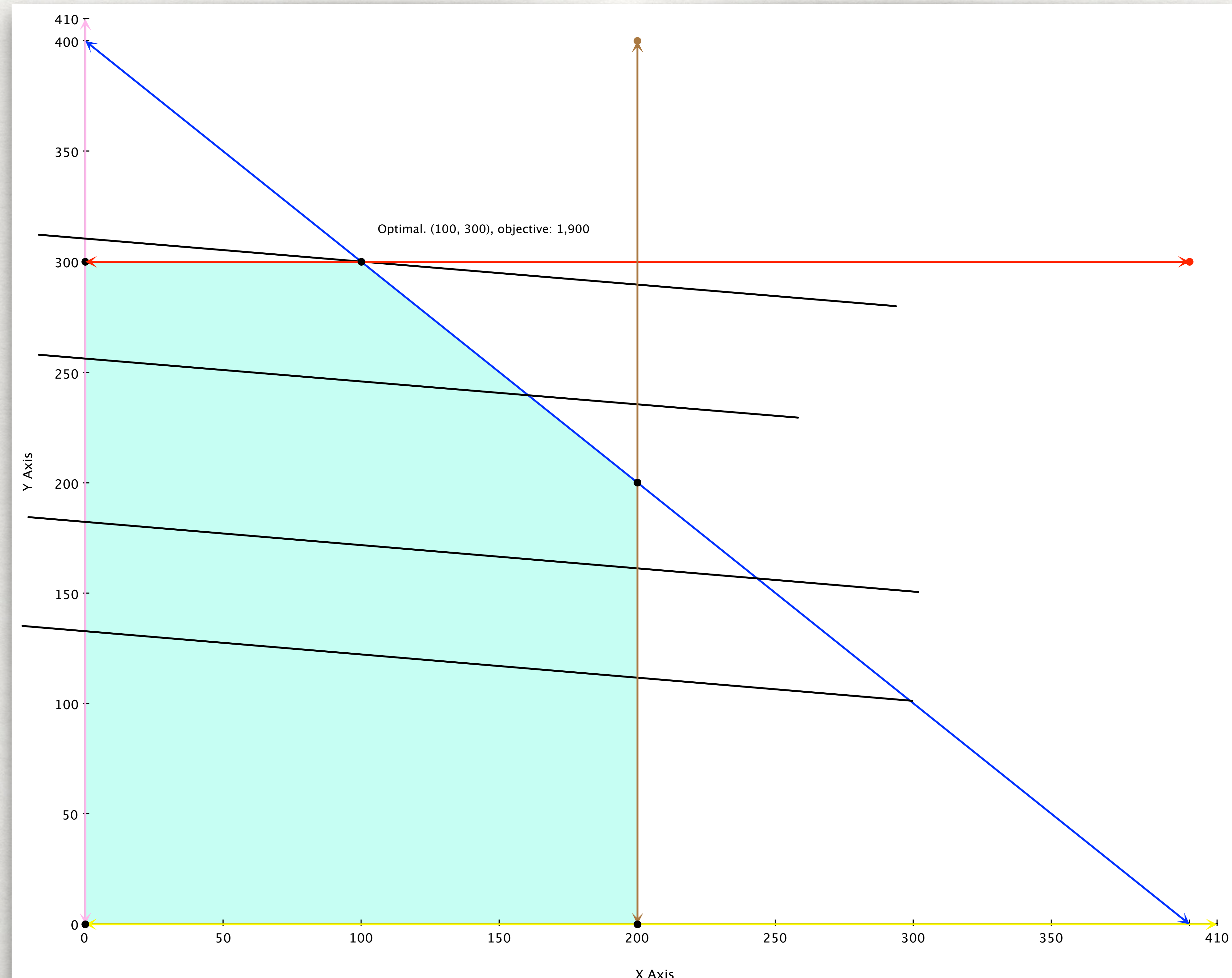
LINEAR PROGRAMMING

- Linear program is infeasible if constraints are too tight. Points cannot simultaneous satisfy all constraints.
    - $x \le 1$ and $x \ge 2$
- Feasible region is unbounded if we can achieve arbitrarily high (low for min problem) objective values.
    - max $x_1 + x_2$ subject to $x_1, x_2 \ge 0$

FEASIBLE REGION

- Feasible region consists of all points that satisfy all constraints.
- Contour lines are objective functions at different intercepts.
- Extreme points are where constraints intersect in feasible region.
- Optimal value where objective function is tangent to the extreme point of feasible region.
  - Otherwise the objective function crosses the feasible region.
  - Can achieve better value of objective function.
  - Feasible region forms convex set.

LINEAR PROGRAMMING



$$\max\ x_1 + 6x_2$$
$$\text{s.t. } x_1 \le 200$$
$$x_2 \le 300$$
$$x_1 + x_2 \le 400$$
$$x_1,\ x_2 \ge 0$$

LINEAR PROGRAMMING

- At optimally, constraints are considered binding if there is no slack.
- LHS = RHS
- Constraints that optimal points lie on.
- From previous page, binding constraints are
  - $x_2 \leq 300$
  - $x_1 + x_2 \leq 400$

DUALITY

- Suppose we have maximization primal LP with optimal value z^*
- How do the constraints affect z^*?
- Suppose we have upper bound of optimal value of LP z_u, $z^* \leq z_u$
- We can minimize z_u until $z^* = z_u$
- Smallest upper bound must be optimal solution to primal LP.
- Weight the constraints to obtain upper bound of objective value.
- Minimize the upper bound (dual LP).
- If primal LP's optimal value is the same as the dual LP's optimal value, we know that the optimal value is indeed the optimal.

DUALITY

Add multipliers y_1, y_2, y_3 for the constraints (not including nonnegativity)

$$\max x_1 + 6x_2$$

s.t.

$$(y_1)\ x_1 \leq 200\ (y_1)$$

$$(y_2)\ x_2 \leq 300\ (y_2)$$

$$(y_3)\ x_1 + x_2 \leq 400\ (y_3)$$

$$x_1 , x_2 \geq 0$$

Suppose $[\ y_1\ y_2\ y_3\ ] = [1, 6, 0]$

$$x_1 + 6x_2 \leq 2000$$

We have an upper bound.

DUALITY

Obtaining linear combination of multipliers and constraints

$$y_1x_1 + y_2x_2 + y_3(x_1 + x_2) \leq 200y_1 + 300y_2 + 400y_3$$

LHS is our upper bound of optimal value.
Rearranging…

$$x_1(y_1 + y_3) + x_2(y_2 + y_3) \leq 200y_1 + 300y_2 + 400y_3$$

To have an upper bound of optimal value, RHS needs to be objective function

$$y_1 + y_3 = 1$$
$$y_2 + y_3 = 6$$

We can relax the constraints to be ≥ as we minimizing over y_i.
Can't be negative because that will give us a trivial optimal of 0.

$$y_1 + y_3 \geq 1$$
$$y_2 + y_3 \geq 6$$

$\max x_1 + 6x_2$
s.t.
$x_1 \leq 200$
$x_2 \leq 300$
$x_1 + x_2 \leq 400$
$x_1, x_2 \geq 0$

DUALITY

Minimize RHS subject to constraints.

$$\min\ 200y_1 + 300y_2 + 400y_3$$

s.t.

$$y_1 + y_3 \geq 1$$

$$y_2 + y_3 \geq 6$$

$$y_1,\ y_2,\ y_3 \geq 0$$

Nonnegativity constraints are added to not change sign of primal LP constraints.

If we have $y_2 < 0$, then inequality sign of constraint 2 in primal LP gets flipped.

$$\max\ x_1 + 6x_2$$

s.t.

$$x_1 \leq 200$$

$$x_2 \leq 300$$

$$x_1 + x_2 \leq 400$$

$$x_1,\ x_2 \geq 0$$

$$\max\ x_1 + 6x_2$$

s.t.

$$(y_1)\ x_1 \leq 200\ (y_1)$$

$$(y_2)\ x_2 \leq 300\ (y_2)$$

$$(y_3)\ x_1 + x_2 \leq 400\ (y_3)$$

$$x_1,\ x_2 \geq 0$$

DUALITY

If we turn constraint 2 into equality, dual becomes

$$\max x_1 + 6x_2$$

s.t.

$$x_1 \leq 200$$
$$x_2 = 300$$
$$x_1 + x_2 \leq 400$$
$$x_1, x_2 \geq 0$$

$$\min 200y_1 + 300y_2 + 400y_3$$

s.t.

$$y_1 + y_3 \geq 1$$
$$y_2 + y_3 \geq 6$$
$$y_1, y_3 \geq 0$$

No nonnegativity for y_2.

$$\max x_1 + 6x_2$$

s.t.

$$(y_1)\ x_1 \leq 200\ (y_1)$$
$$(y_2)\ x_2 = 300\ (y_2)$$
$$(y_3)\ x_1 + x_2 \leq 400\ (y_3)$$
$$x_1, x_2 \geq 0$$

DUALITY

Primal LP

max $x_1 + 6x_2$

s.t.

$x_1 \leq 200$

$x_2 \leq 300$

$x_1 + x_2 \leq 400$

$x_1, x_2 \geq 0$

$$\begin{aligned} &\max \mathbf{c^T x} \\ \text{s.t.} \quad &\mathbf{Ax} \leq \mathbf{b} \\ &\mathbf{x} \geq \mathbf{0} \end{aligned}$$

Dual LP

min $200y_1 + 300y_2 + 400y_3$

s.t.

$y_1 + y_3 \geq 1$

$y_2 + y_3 \geq 6$

$y_1, y_2, y_3 \geq 0$

$$\begin{aligned} &\min \mathbf{y^T b} \\ \text{s.t.} \quad &\mathbf{y^T A} \geq \mathbf{x} \\ &\mathbf{y^T} \geq \mathbf{0^T} \end{aligned}$$

$$\max \begin{bmatrix} 1 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$
$$\text{s.t.} \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \leq \begin{bmatrix} 200 \\ 300 \\ 400 \end{bmatrix}$$
$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \geq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\min \begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix} \begin{bmatrix} 200 \\ 300 \\ 400 \end{bmatrix}$$
$$\text{s.t.} \quad \begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \geq \begin{bmatrix} 1 & 6 \end{bmatrix}$$
$$\begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix} \geq \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$$

SOLVING LINEAR PROGRAMS

1. Graph feasible regions and find tangency between objective function and extreme points.

2. Find extreme points by intersecting constraints. Try extreme points in objective function.

3. Solve the dual LP. Plug in y_i multipliers and objective value into primal LP and solve for unknowns (x_i) using algebra.