CS 170 DISCUSSION 10

MAXIMUM FLOW

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UC Berkeley Fall 17

MAXIMUM FLOW

- Given a directed graph G = (V, E), send as many units of flow from source node *s* to sink node *t*.
- Edges have capacity constraints.
- When sending flow through some path, flow must be at most the capacity constraint of every edge.
- Number of units sent from *s* = number of units received by *t*.
- Applications: supply chain shipping, network flows, supply and demand shipping (need modification).

MAXIMUM FLOW

Max Flow Linear Program

$$\text{Maximize} \quad \text{size}(f) = \sum_{(s,u)\in E} f_{su}$$

$$\text{Subject to} \quad \sum_{(w,u)\in E} f_{wu} = \sum_{(u,v)\in E} f_{uv} \quad \forall u \in V \quad \text{Flow conservation consraints}$$

$$0 \leq f_e \leq c_e \quad \forall e \in E \quad \text{Capacity and nonnegativity constraints}$$

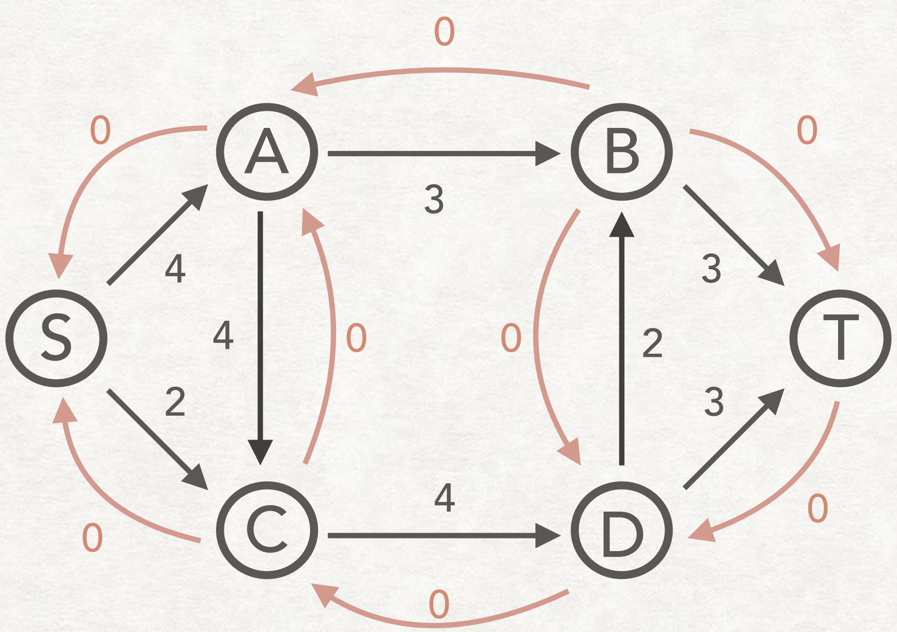
RESIDUAL GRAPH

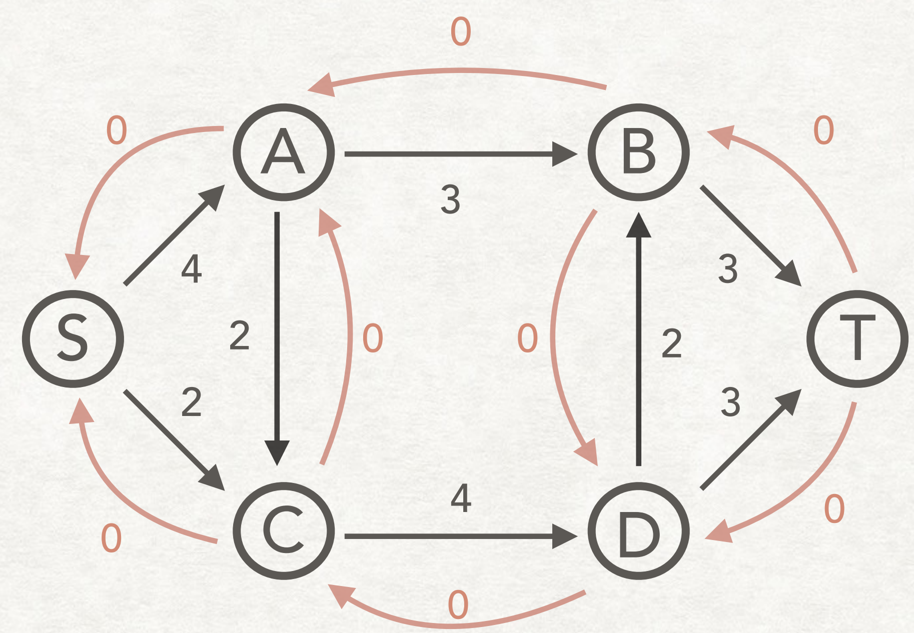
- When sending f_{uv} through edge (u, v), we create a residual back edge (v, u).
- Capacity of (u, v) = $c_{uv} - f_{uv}$
- Capacity of (v, u) = f_{uv}
- Keep sending flows from *s* to *t*.
- Once we have no path from *s* to *t*, we have the maximum flow.
- Edges can be removed if capacity reaches 0. This will ensure we won't have a path eventually.
- Can use residual edges later to go back on previous flow decisions.

MAX FLOW ALGORITHMS

- Ford-Fulkerson algorithm.
  - In each iteration, send as many units of flow as possible and build residual graph.
  - Stop condition: no *s*-*t* path in residual graph.
- Edmond-Karp algorithm.
  - Find shortest *s*-*t* path via BFS in terms of edge hops.
  - Send as many units of flow as possible and build residual graph until no *s*-*t* path.
  - If capacities are integral, solution flows are integral, runtime is $O(|V||E|^2)$.

MAX FLOW RESIDUAL GRAPH EXAMPLE





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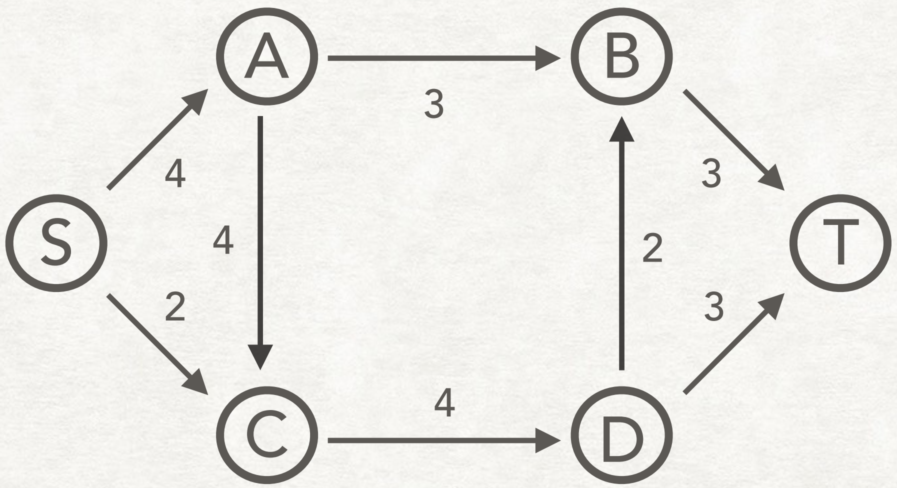
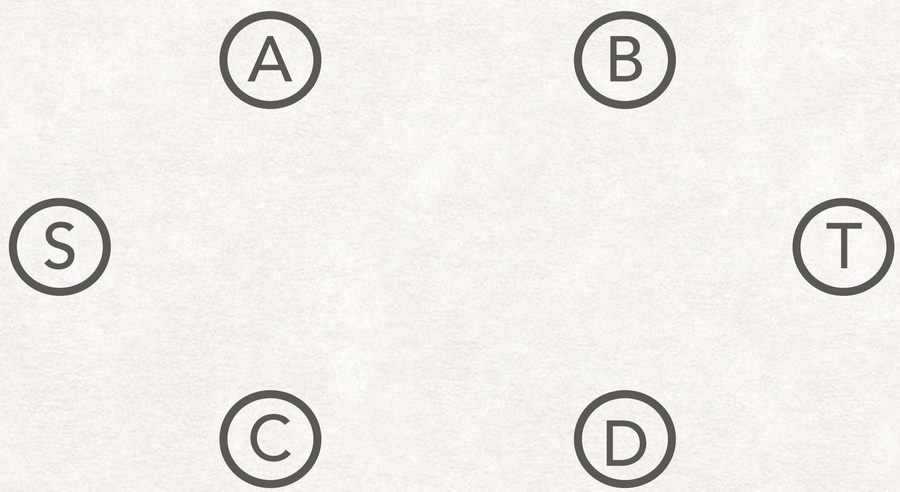
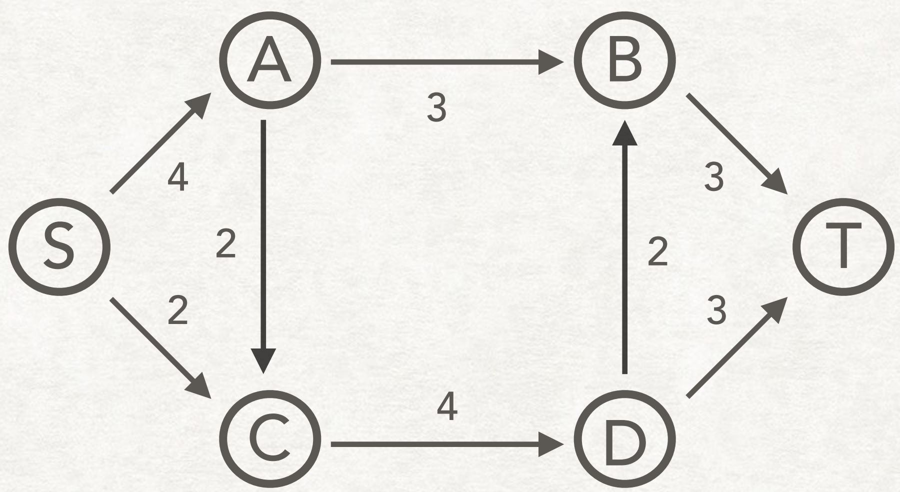


MAX FLOW RESIDUAL GRAPH EXAMPLE

Flow graph



Residual graph



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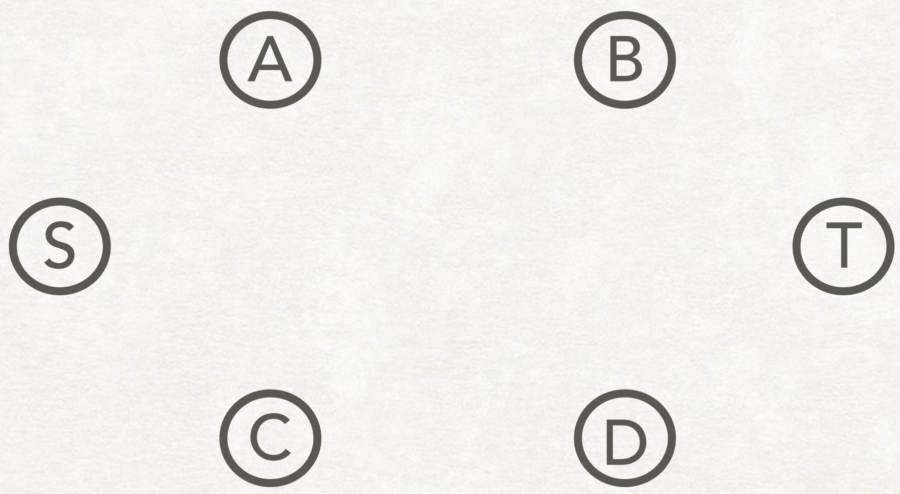

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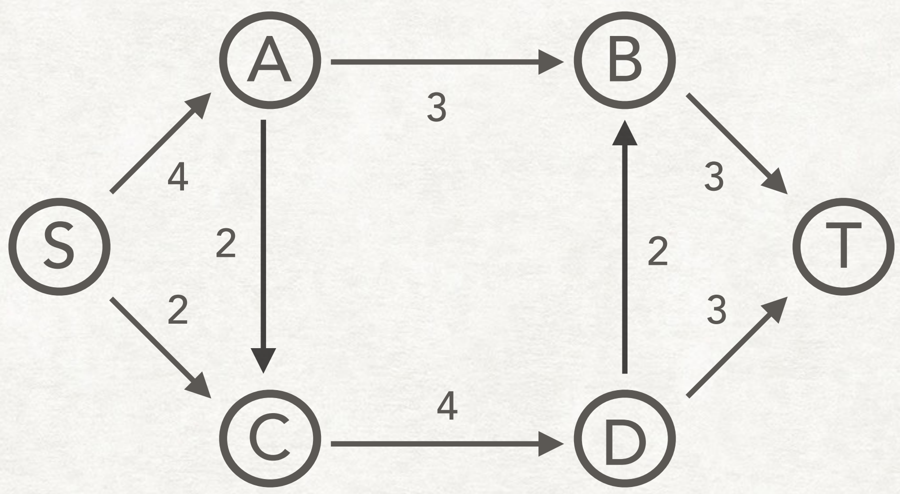
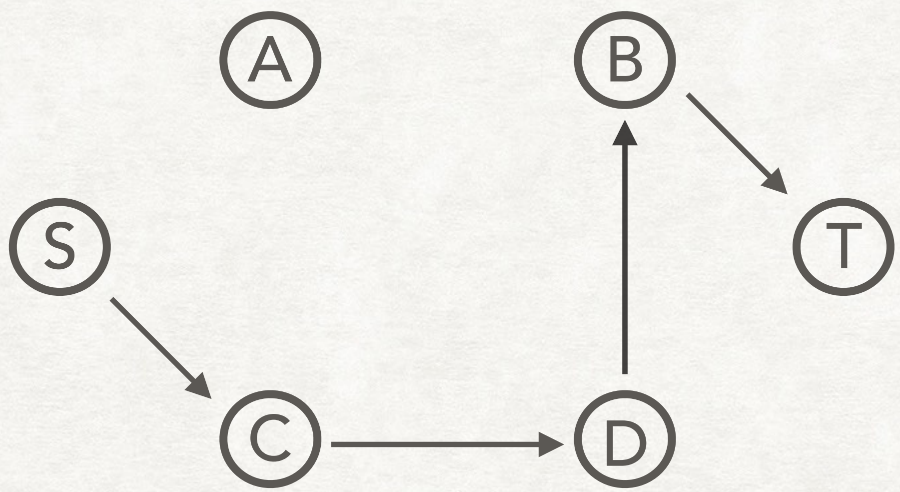


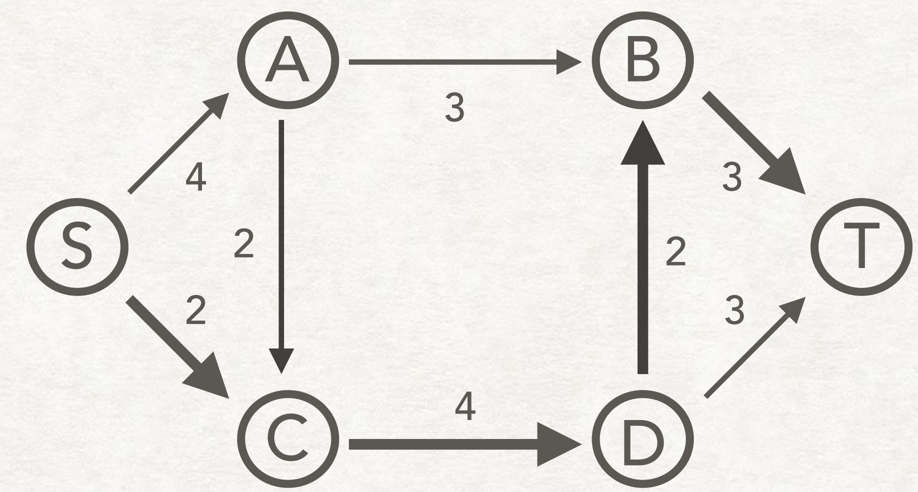
Suppose we send units through SCDBT first.

MAX FLOW RESIDUAL GRAPH EXAMPLE

Flow graph



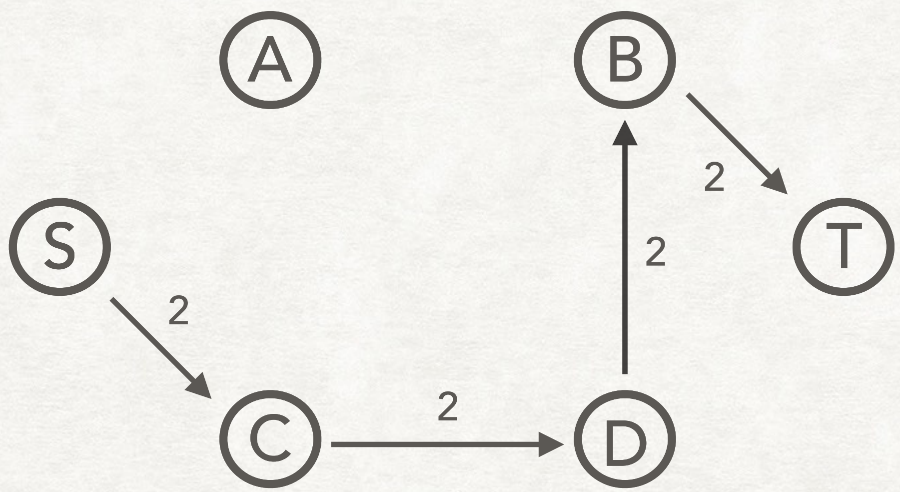
Residual graph



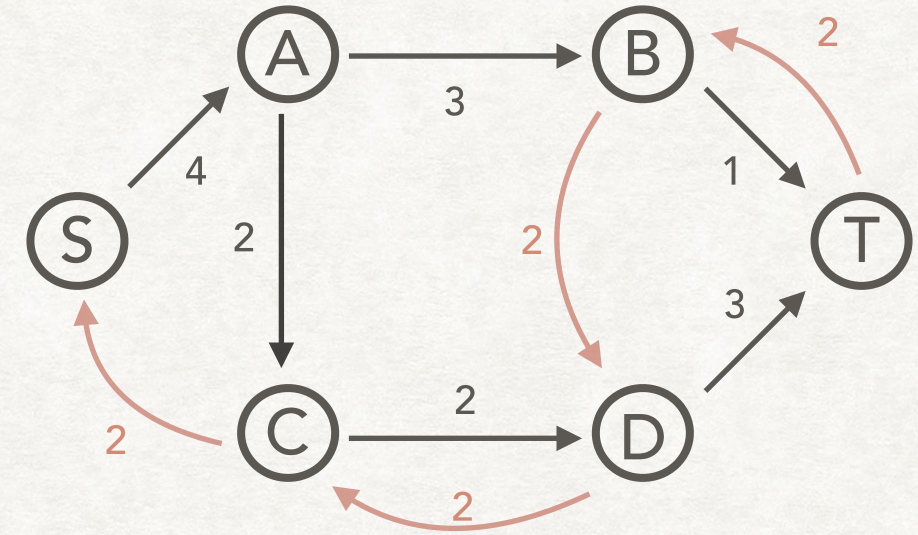
Send 2 units via SCDBT.

MAX FLOW RESIDUAL GRAPH EXAMPLE

Flow graph



Residual graph



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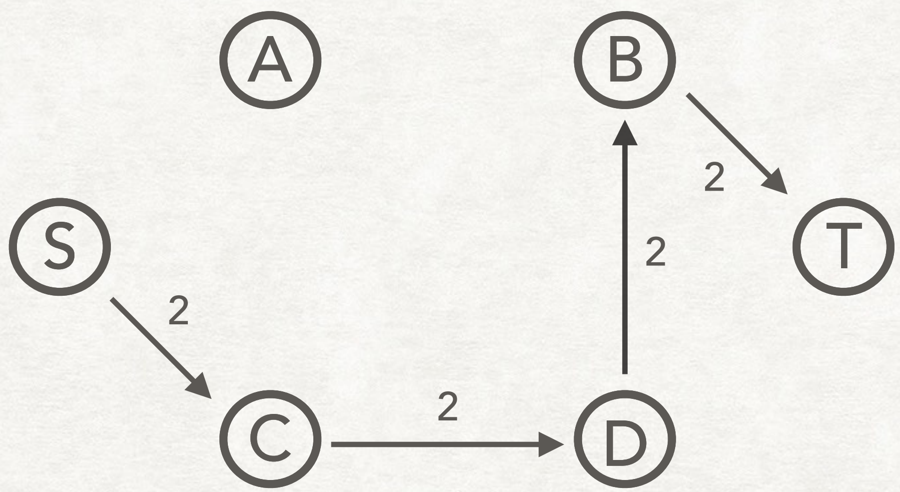

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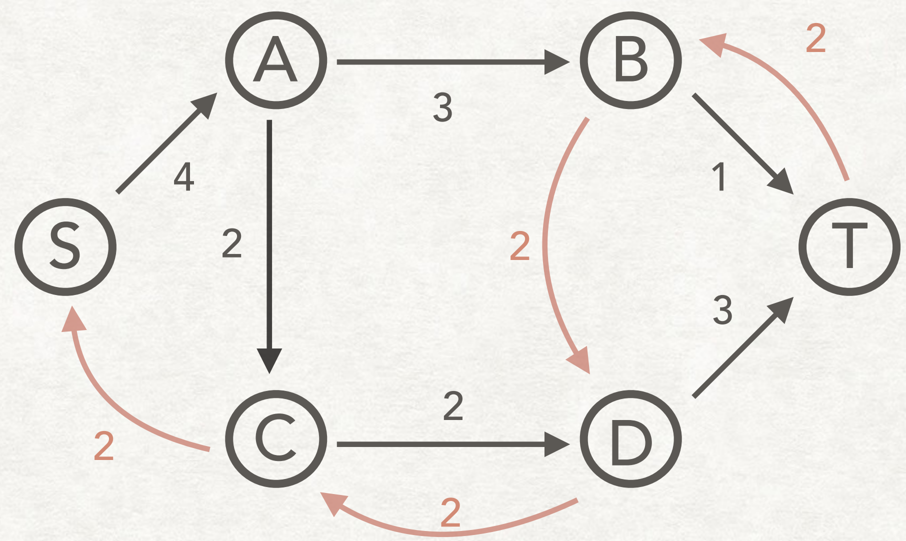

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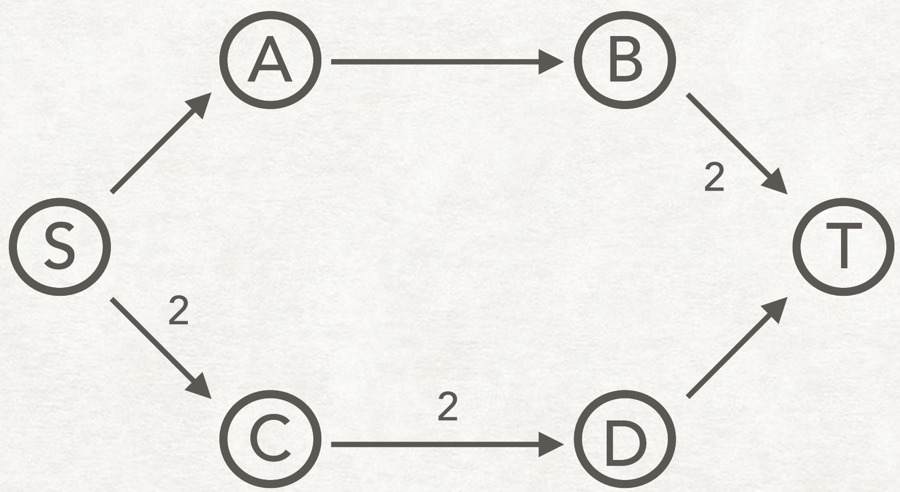

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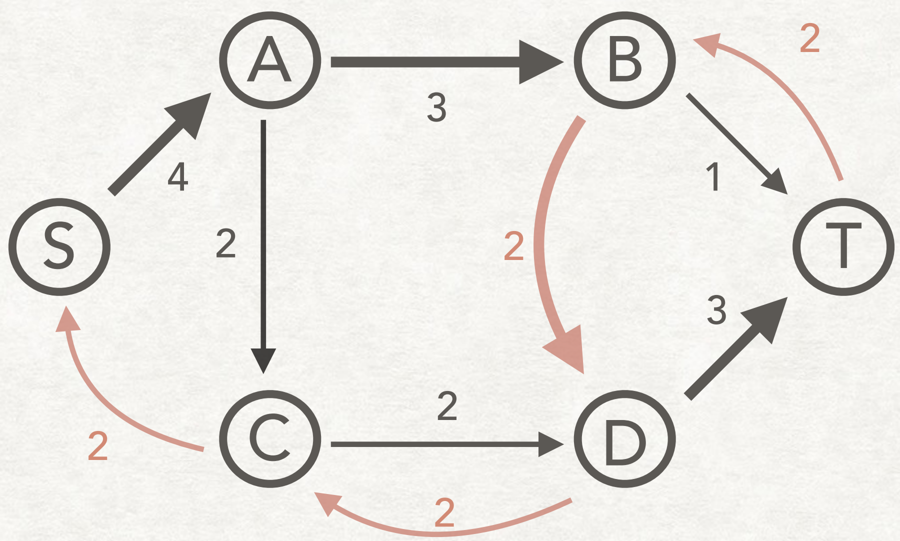

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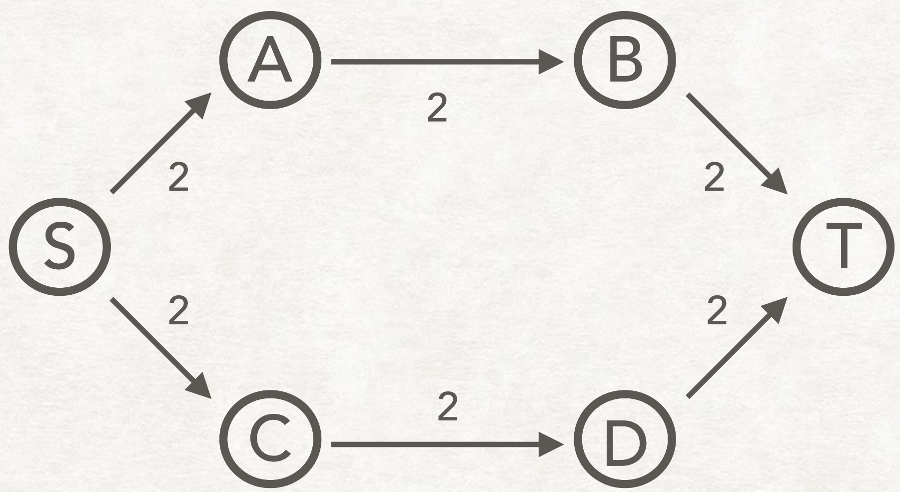

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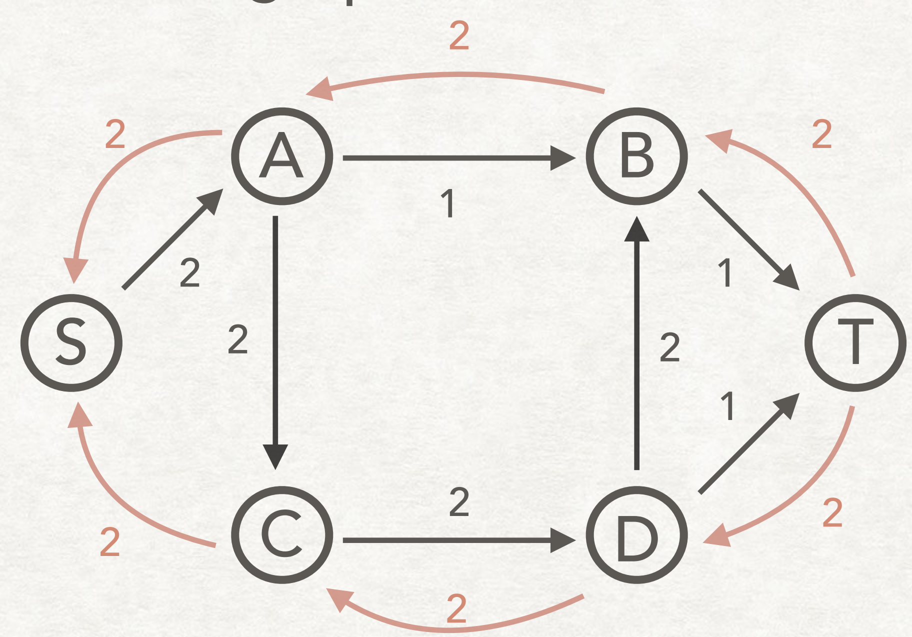

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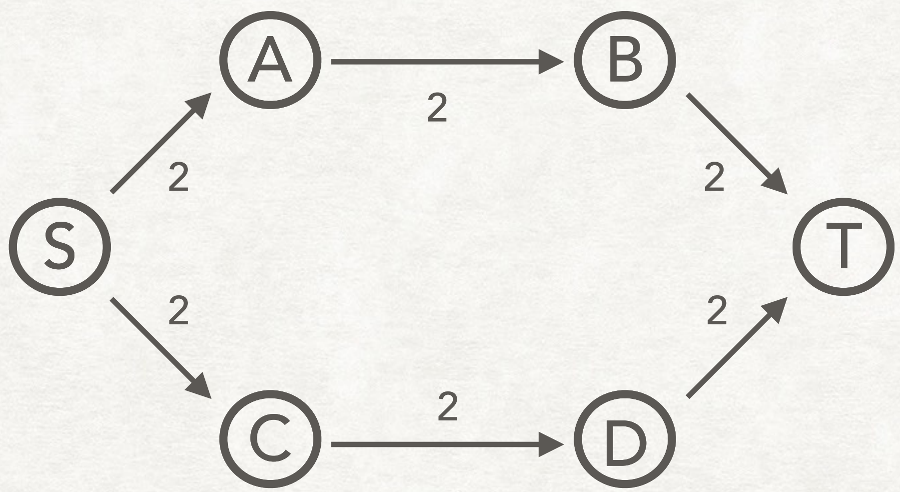

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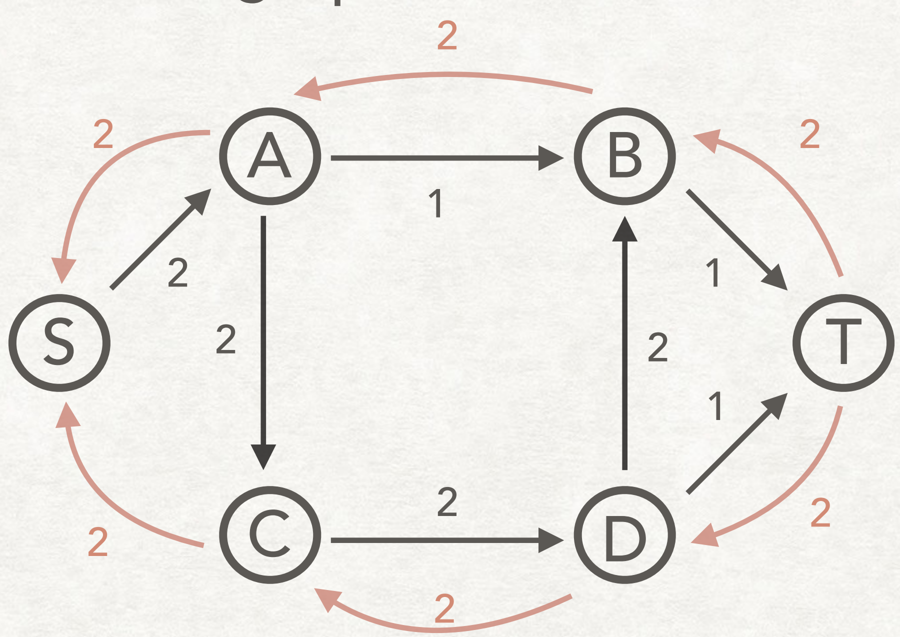

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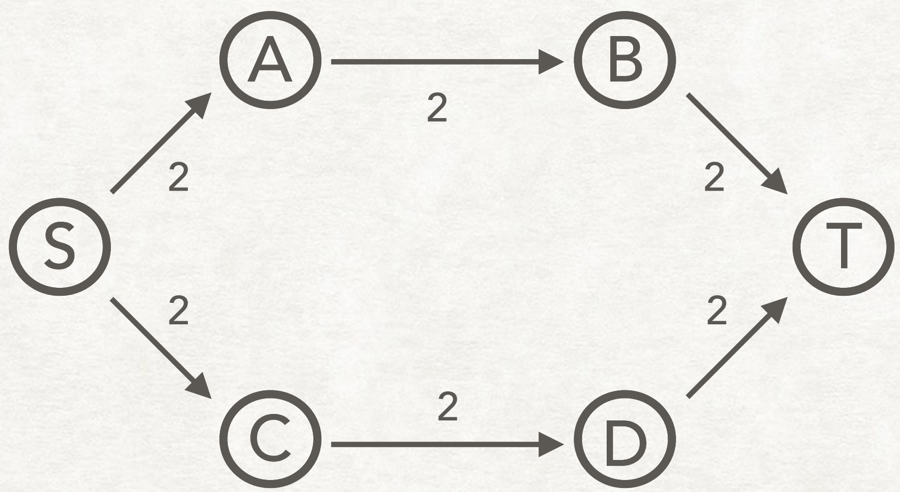

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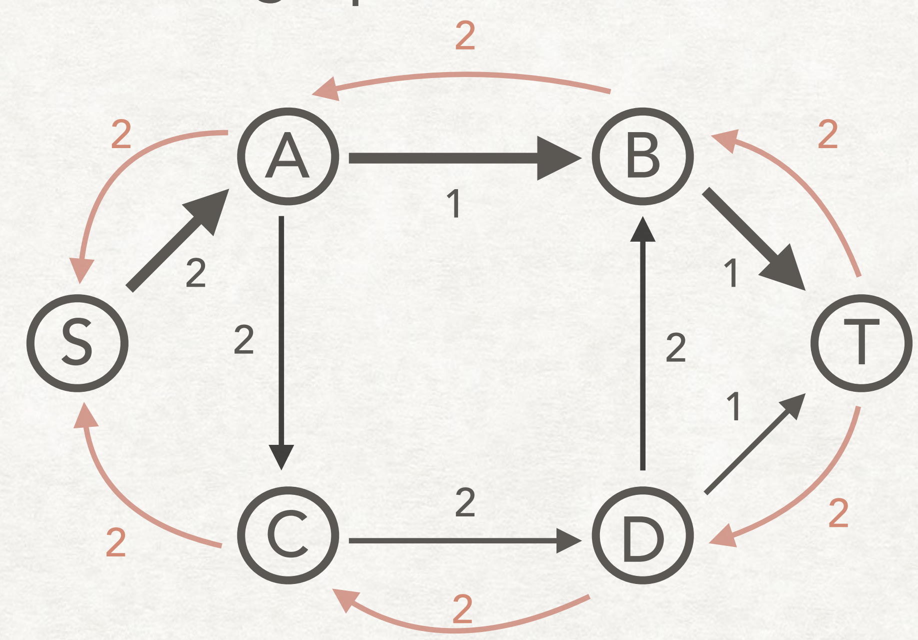

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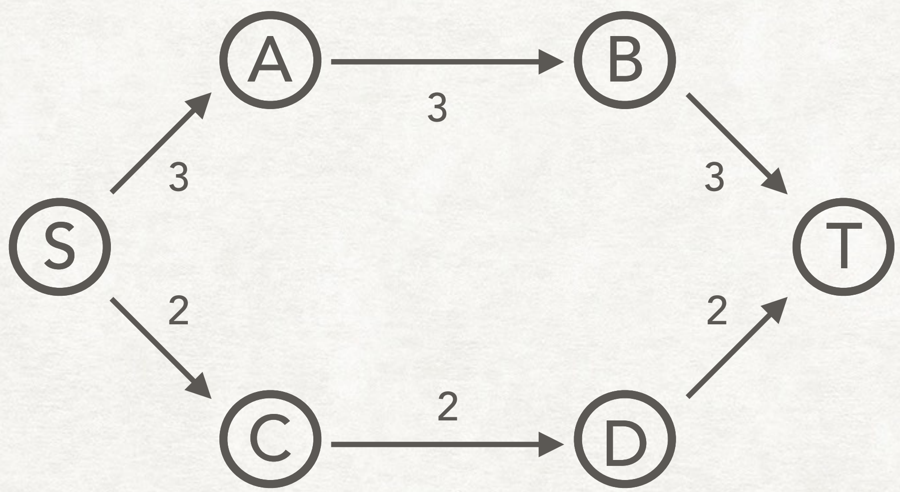

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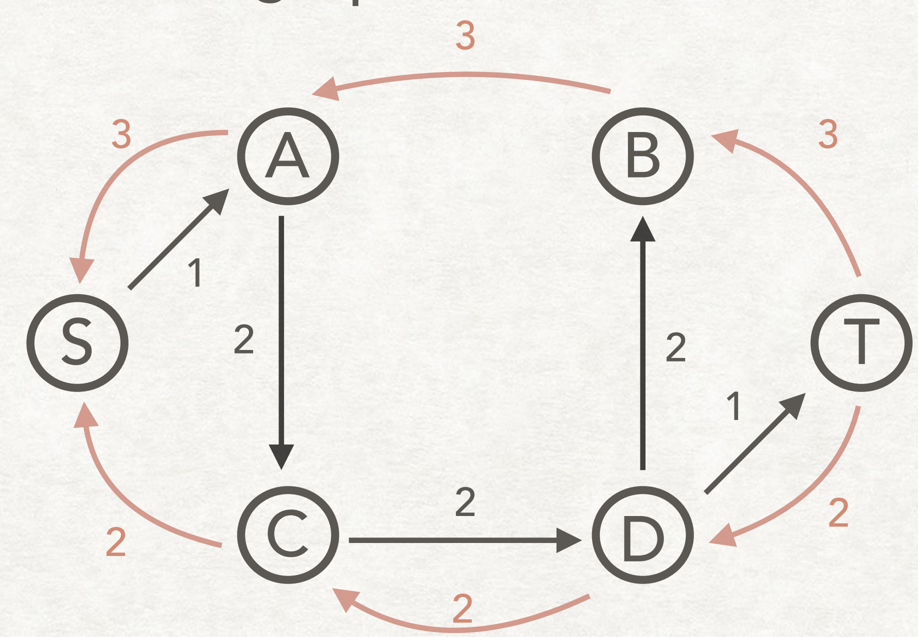

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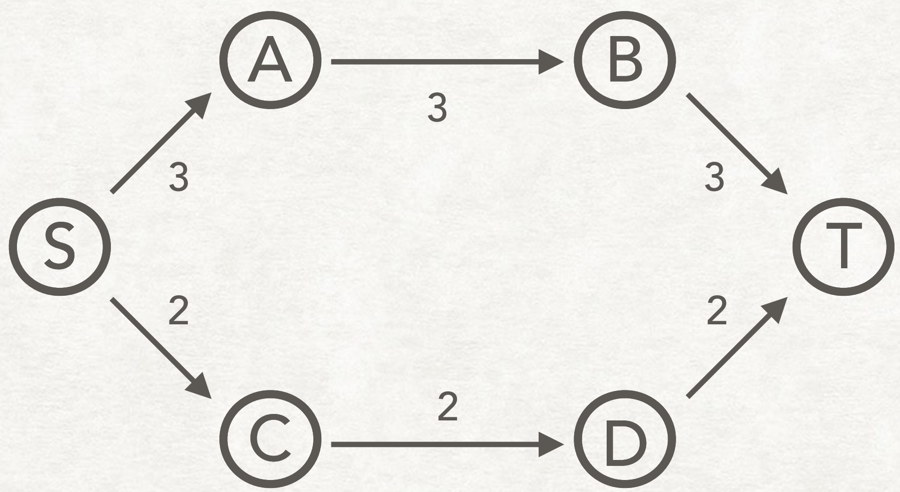

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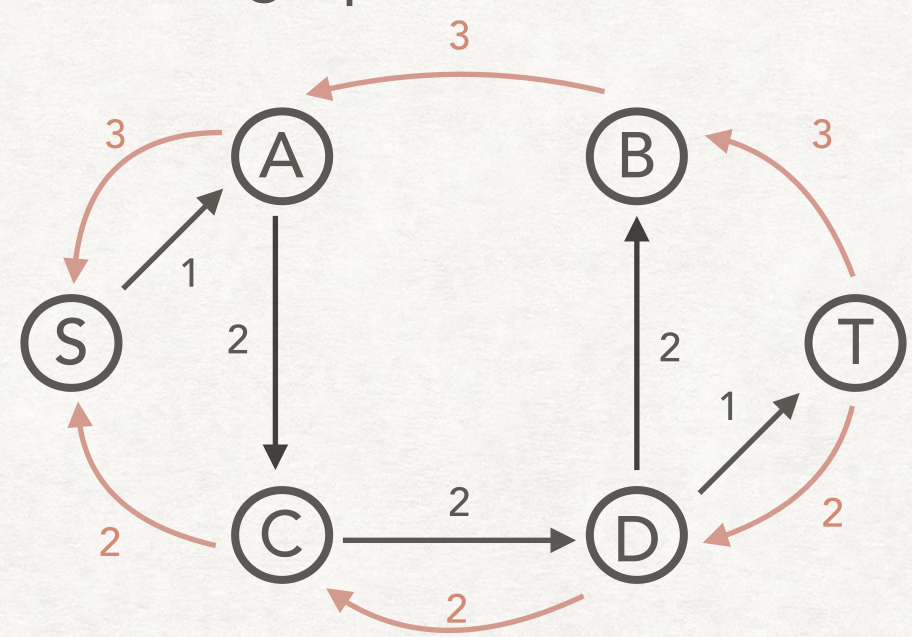

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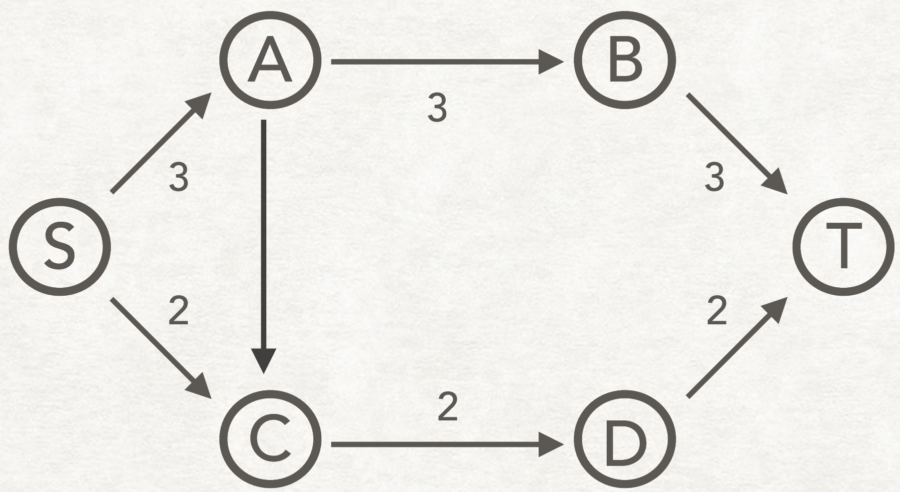

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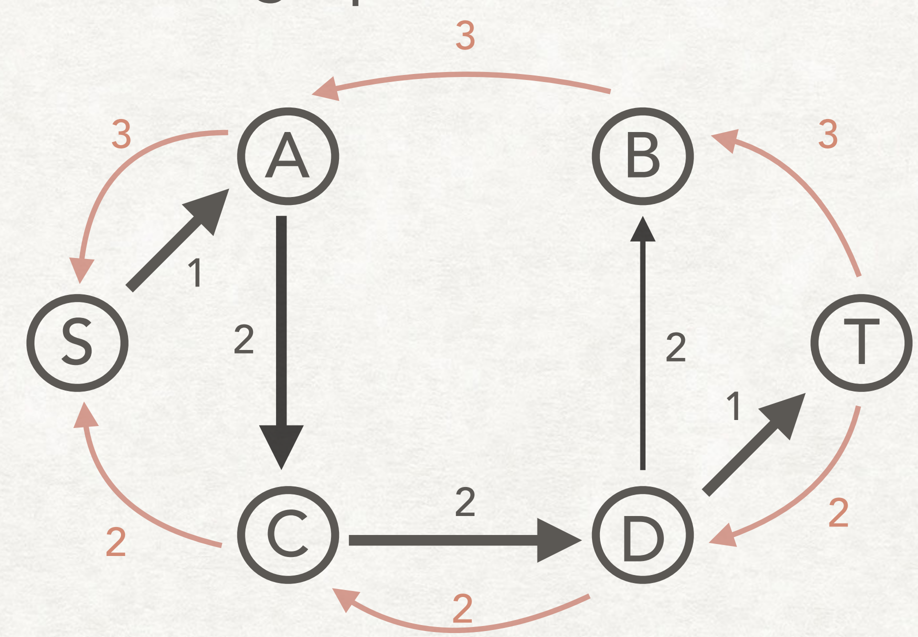

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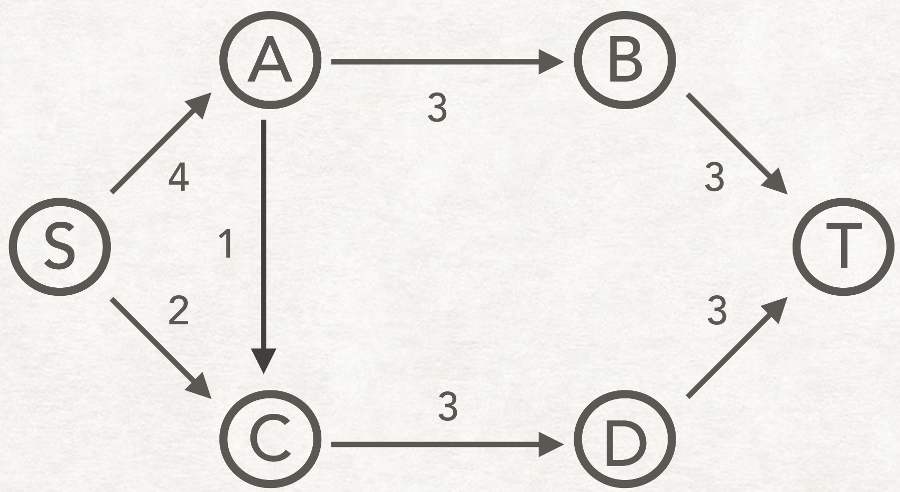

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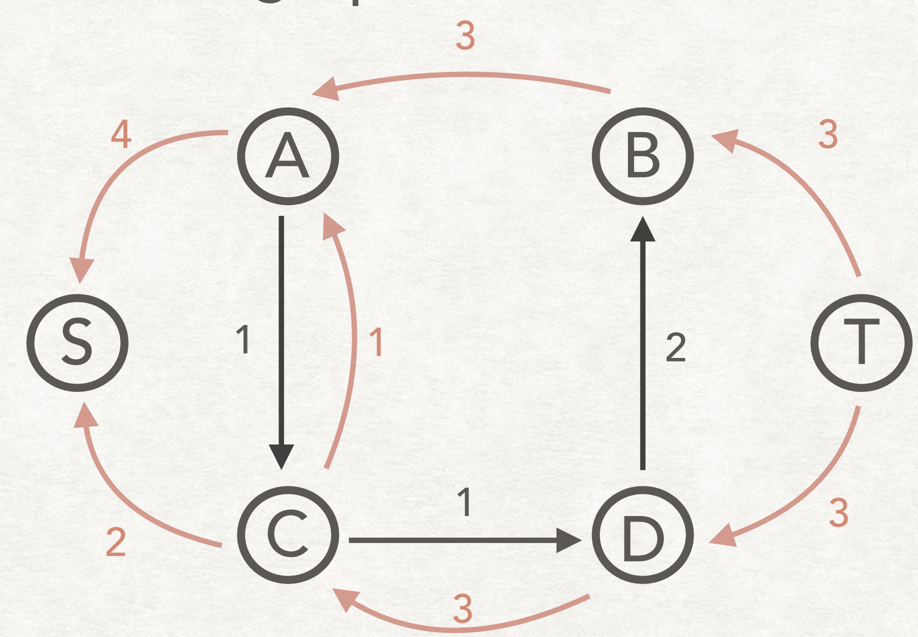

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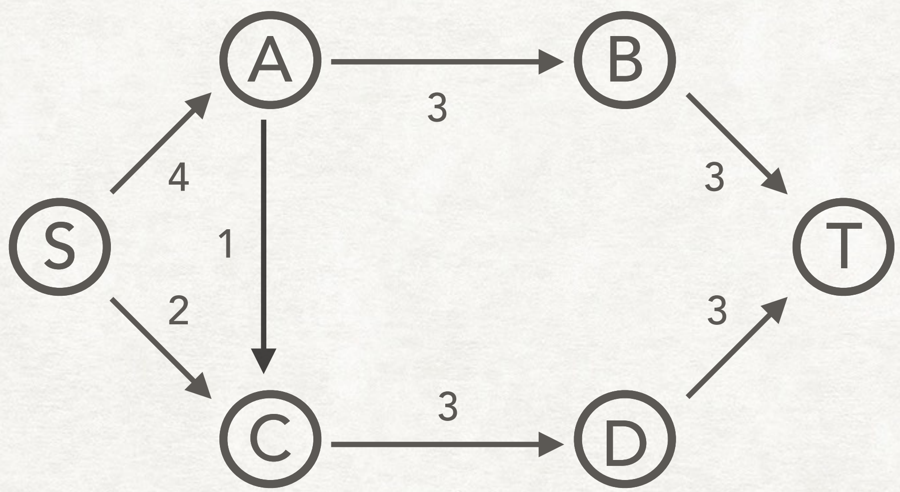

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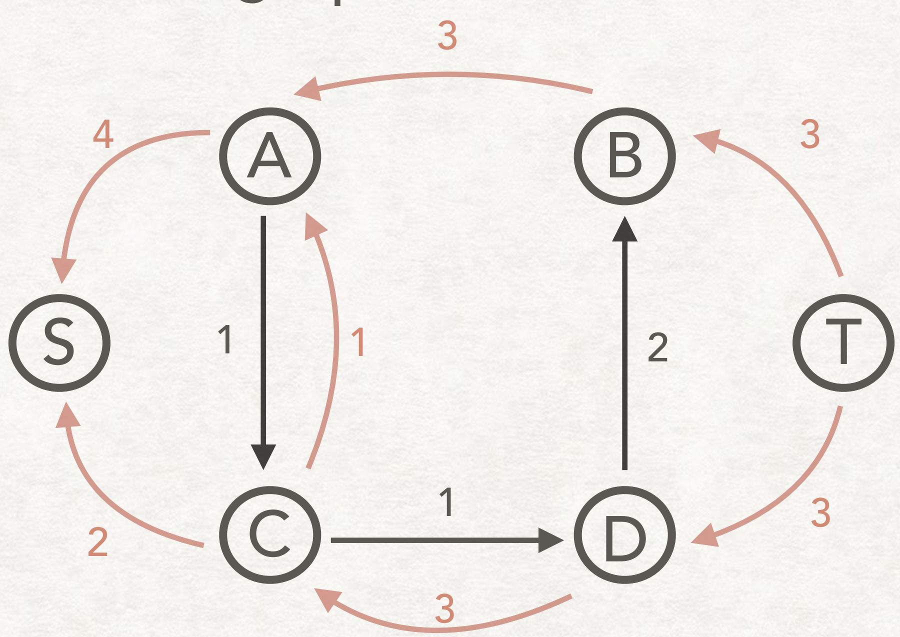

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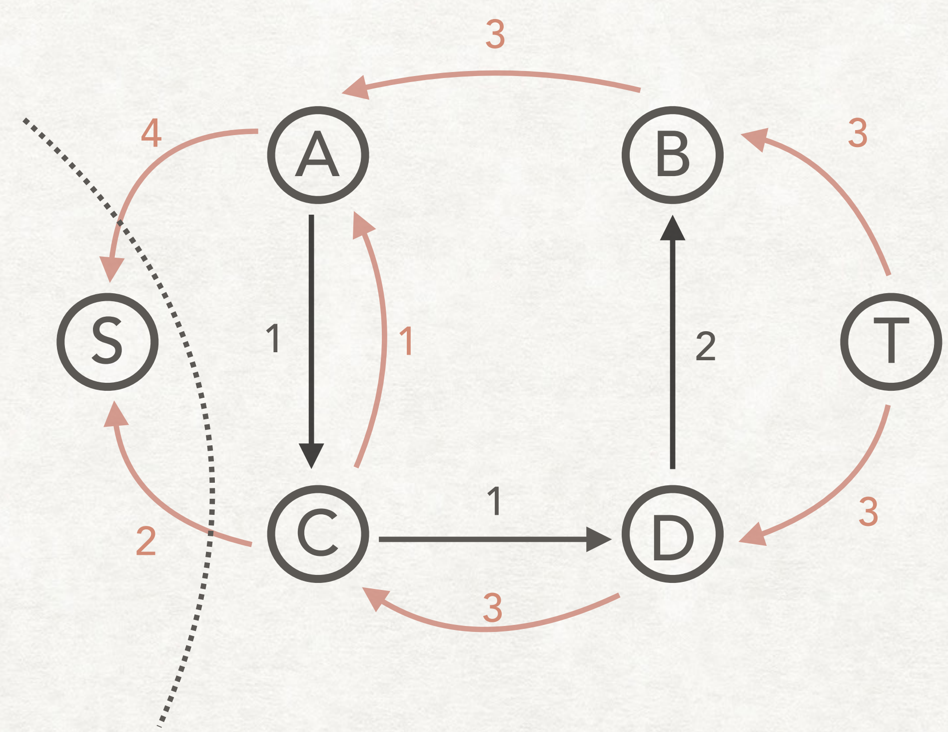
No more *s-t* path. Maximum flow is 6.
After computing max flow, output edges with their respective flows (see flow graph).

MINIMUM CUT

- A cut partitions the vertices into two disjoint sets L and R.
- The capacity of a cut is the sum of edges that crosses the cut (go from L to R).
- A (*s*, *t*)-cut partitions graph such that L contains *s* and all vertices *s* can reach and R contain *t* and all vertices that can reach *t*.
- For any flow *f* and any (*s*, *t*)-cut (L, R), size(*f*) ≤ capacity(L, R).
- The capacity of the cuts provide an upper bound of the maximum flow.
- Size of maximum flow equals to capacity of smallest *s*-*t* cut.

MINIMUM CUT

- {S} and {A, C, B, D, T} forms a min cut.
- {S, A, B, C, D} and {T} forms another min cut.
- Minimum cut will contain only edges in the residual graph that goes from R to L.
- In the original graph, only edges from L to R in the minimum cut.
- These edges are fully saturated (at full capacity).



REDUCTIONS

- When using Max Flow to solve problems, make use of reduction.
- Reduction: solve problem A by solving problem B.
  - Suppose we have a problem A,
  - Preprocess input to fit inputs for problem B (suppose max flow).
  - Solve problem B.
  - Postprocess output of problem B to fit output of problem A.
- Example. Use Max flow to solve network flows with supply and demand nodes.