CS 170 DISCUSSION 11

ZERO-SUM GAMES

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ZERO-SUM GAMES

- Two players play a game.
- To maximize each's own score, need to minimize other player's score.
- Each player has a set of actions.
  - Note that they need not have the same number of actions (i.e. player A has 3 actions while player B has 2 actions).
- Each player's action will add x points to a player's score and subtract x points from a player's score.
  - Points are with respective to **one** player.

ZERO-SUM GAMES

- If player A always makes the same move (pure strategy), then player B will optimize his/her strategy against that move.
- Thus both players want to vary their actions with respective to some probability distribution (mix strategy).
- Want to find the best probability distribution to maximize own score (or minimize opponent score).
- User payoff matrix.

ZERO-SUM GAMES

Column player

Payoffs with respective to row player

Row player

	C_1	C_2
R_1	3	-1
R_1	-2	1

ZERO-SUM GAMES

- Let x_1, x_2 by probability that row player chooses actions R_1, R_2 respectively.
- Let y_1, y_2 by probability that row player chooses actions C_1, C_2 respectively.
- Suppose row announces $\mathbf{x} = (x_1, x_2)$, column has pure strategies of $\mathbf{y} = (1, 0)$ or $(0, 1)$ and will minimize the expected score (row player payoffs).
- Column can give row score of either
  - $3x_1 - 2x_2$
  - $-x_1 + x_2$

	C_1	C_2
R_1	3	-1
R_1	-2	1

ZERO-SUM GAMES

- Row objective to max (min { $3x_1 - 2x_2$, $-x_1 + x_2$ })
- Let z = min { $3x_1 - 2x_2$, $-x_1 + x_2$ }
  - z is upper bounded by equations.
  - Smaller value will be binding.
  - By upper bounded z, we effectively choose z to be the minimum of the two.
  - $z \leq 3x_1 - 2x_2$
  - $z \leq -x_1 + x_2$
  - z is the outcome of Column's best response to **x**

	C_1	C_2
R_1	3	-1
R_1	-2	1

ZERO-SUM GAMES

- Following LP for Row player
- Ensures **x** is a proper probability distribution with last equality and nonnegativitiy constraints.

	C_1	C_2
R_1	3	-1
R_1	-2	1

$$
\begin{array}{rcrcrcl}
 & & \max & & z & & \\
-3x_1 & + & 2x_2 & + & z & \leq & 0 \\
x_1 & - & x_2 & + & z & \leq & 0 \\
x_1 & + & x_2 & & & = & 1 \\
 & & & & x_1, x_2 & \geq & 0
\end{array}
$$

ZERO-SUM GAMES

- Similar logic from column's perspective. But we want to minimize because payoffs are for row.
- Column choose **y** = (y_1, y_2) that minimizes the best outcome that Row can give him/her.
- min (max { $3y_1 - y_2$, $-2y_1 + y_2$ })

$$
\begin{array}{rcl}
 & \min & w \\
-3y_1 + y_2 + w & \geq & 0 \\
2y_1 - y_2 + w & \geq & 0 \\
y_1 + y_2 & = & 1 \\
y_1, y_2 & \geq & 0
\end{array}
$$

	C_1	C_2
R_1	3	-1
R_1	-2	1

ZERO-SUM GAMES

- The following LPs are **duals** of each other. (Try the math).

$$\begin{array}{rcrcrcl} & & \max & & z & & \\ -3x_1 & + & 2x_2 & + & z & \leq & 0 \\ x_1 & - & x_2 & + & z & \leq & 0 \\ x_1 & + & x_2 & & & = & 1 \\ & & & & x_1, x_2 & \geq & 0 \end{array}$$

$$\begin{array}{rcrcrcl} & & \min & & w & & \\ -3y_1 & + & y_2 & + & w & \geq & 0 \\ 2y_1 & - & y_2 & + & w & \geq & 0 \\ y_1 & + & y_2 & & & = & 1 \\ & & & & y_1, y_2 & \geq & 0 \end{array}$$

	C_1	C_2
R_1	3	-1
R_1	-2	1

- They have the same optimal solution and optimal value V.

- No matter who declares his/her mixed strategy first, they will end up with the same optimal score, if both play optimally.

ZERO-SUM GAMES

- The duality can be generalized to arbitrary games.
- There exists optimal strategies for both players and achieve the same value.
- In game theory, it's called the min-max theorem.

	C_1	C_2
R_1	3	-1
R_1	-2	1

$$\max_{\mathbf{x}} \min_{\mathbf{y}} \sum_{i,j} G_{ij} x_i y_j = \min_{\mathbf{y}} \max_{\mathbf{x}} \sum_{i,j} G_{ij} x_i y_j.$$