CS 170 Spring 2017 − Discussion 2

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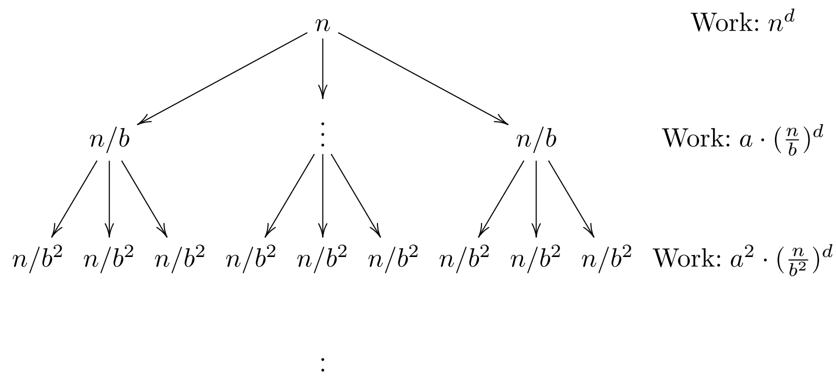
Master's Theorem

For any recurrence relation with the following structure,

$$T(n) = a \cdot T(\lceil n/b \rceil) + O(n^d)$$

$$T(n) = \begin{cases} O(n^d) & \text{if } d > \log_b a \\ O(n^d \log n) & \text{if } d = \log_b a \\ O(n^{\log_b a}) & \text{if } d < \log_b a \end{cases}$$

If we take a look at the following unrolled recurrence tree.



At some k-th level, the size of the subproblem is n/b^k. Because of the branching factor of a, there are a^k of these subproblems and the amount of work at that level will be $O(a^k \cdot (\frac{n}{b^k})^d)$.

$$\begin{aligned} a^k \cdot (\frac{n}{b^k})^d &= n^d \cdot \frac{a^k}{b^{kd}} \\ &= n^d \cdot (\frac{a}{b^d})^k \end{aligned}$$

The total amount of work for the entire recurrence tree is the sum of the work at each level. Since each subproblem gets smallers by factor $1/b$, there are a total $\log_b n$ levels. Summing up the terms gives us the following geometric series.

$$T(n) = \sum_{k=0}^{\log_b n} O(n^d) \cdot (\frac{a}{b^d})^k$$

Let's first look at the sum of a geometric series.

$$\begin{aligned} S(n) = \sum_{k=0}^{n} ar^k &= a\frac{r^{n+1} - 1}{r - 1}, \text{ if } r \neq 1 \\ &= a\frac{r \cdot r^n - 1}{r - 1} \\ &= a\Big(r^n \frac{r}{r-1} - \frac{1}{r-1}\Big) \end{aligned}$$

If $r > 1$, we have an increasing geometric series. When n gets significantly large, r^n approaches infinity, and

$$\begin{aligned} &S(n) < ar^n \frac{r}{r-1} \\ &S(n) \leq c \cdot \frac{r}{r-1} \cdot ar^n \\ &S(n) \in O(ar^n) \end{aligned}$$

Also,

$$
\begin{aligned}
S(n) &= a\Big(r^n \frac{r}{r-1} - \frac{1}{r-1}\Big) \\
S(n) &> ar^n \\
S(n) &\in \Omega(ar^n)
\end{aligned}
$$

Thus when $r > 1$, $S(n) \in O(ar^n), \Omega(ar^n), \Theta(ar^n)$.

The sum can also be written as

$$
\begin{aligned}
S(n) = \sum_{k=0}^{n} ar^k &= a\frac{1 - r^{n+1}}{1-r}, \text{ if } r \neq 1 \\
&= a\Big(\frac{1}{r-1} - r^n \frac{r}{r-1}\Big)
\end{aligned}
$$

Now if $r < 1$, we have a decreasing geometric series. When n get significantly large, r^n approaches 0, and

$$
\begin{aligned}
S(n) &< a\frac{1}{r-1} \\
S(n) &\leq c \cdot a \\
S(n) &\in O(a)
\end{aligned}
$$

Also,

$$
\begin{aligned}
S(n) &= a\Big(\frac{1}{r-1} - r^n \frac{r}{r-1}\Big) \\
S(n) &\geq c \cdot a \\
S(n) &\in \Omega(a)
\end{aligned}
$$

Thus when $r < 1, S(n) \in O(a), \Omega(a), \Theta(a)$.
When $r = 1$, all the terms in the series is a. Thus $S(n) = an$ and $S(n) \in \Theta(an)$.

Substituting $O(n^d)$ as a, r as $\frac{a}{b^d}$, and n as $\log_b n$, we get back our geometric sum of the recurrence tree.
For a decreasing geometric series,

$$
\begin{aligned}
r = \frac{a}{b^d} &< 1 \\
a &< b^d \\
\log_b a &< d
\end{aligned}
$$

$S(n) \in \Theta(a)$ and $T(n) \in \Theta(n^d)$, which is the first case of the master's theorem. Note that we prove it for $\Theta(\cdot)$, which means it works for both $O(\cdot)$ and $\Omega(\cdot)$.

When

$$
\begin{aligned}
r = \frac{a}{b^d} &= 1 \\
\log_b a &= d
\end{aligned}
$$

$S(n) \in \Theta(an)$ and $T(n) \in \Theta(n^d \log_b n)$, which is the second case of the master's theorem.

For an increasing geomteric series,

$$
\begin{aligned}
r = \frac{a}{b^d} &> 1 \\
\log_b a &> d
\end{aligned}
$$

$S(n) \in \Theta(ar^n)$ and $T(n) \in \Theta(n^d \cdot \left(\frac{a}{b^d}\right)^{\log_b n})$

$$\begin{aligned}
n^d \cdot \left(\frac{a}{b^d}\right)^{\log_b n} &= n^d \cdot \frac{a^{\log_b n}}{(b^d)^{\log_b n}} \\
&= n^d \cdot \frac{n^{\log_b a}}{n^{\log_b b^d}} \\
&= n^d \cdot \frac{n^{\log_b a}}{n^{d \log_b b}} \\
&= n^d \cdot \frac{n^{\log_b a}}{n^{d \cdot 1}} \\
&= n^{\log_b a}
\end{aligned}$$

Thus $T(n) \in \Theta(n^{\log_b a})$, which is the third case of the master's theorem.