CS 170 Spring 2017 − Discussion 3

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Fast Fourier Transform

Polynomial Multiplication

Given two polynomials $A(x) = a_0 + a_1x + a_2x^2 + \dots a_dx^d$ and $B(x) = b_0 + b_1x + b_2x^2 + \dots b_dx^d$, we want $C(x) = A(x) \cdot B(x) = c_0 + c_1x + c_2 + x^2 + \cdots + c_{2d}x^{2d}$ where

$$c_k = a_0b_k + a_1b_{k-1} \dots a_kb_0 = \sum_{i=0} ka_ib_{k-i}$$

This is really slow because we have to evaluate every pairwise coefficients between $A(x)$ and $B(x)$ to compute $C(x)$, which is $O(d^2)$.

Since any polynomial with degree d can be determined by $d + 1$ points, we can use these values to represent our polynomials. Now $C(x_i) = A(x_i) \cdot B(x_i)$. The step would take only $O(d)$. Below we have another method for polynomial multiplication.

- **Selection**
  Pick points $x_0, x_1, \dots, x_{n-1}$, $n \geq 2d + 1$.
- **Evaluation**
  Compute $A(x_0), A(x_1), \dots, A(x_{n-1}), B(x_0), B(x_1), \dots, B(x_{n-1})$.
- **Multiplication**
  Compute $C(x_k) = A(x_k) \cdot B(x_k)$, $k = 0, 1, \dots, n - 1$.
- **Interpolation**
  Recover $C(x) = c_0 + c_1x + c_2x^2 + \dots c_{2d}x^{2d}$ from $C(x_k)$, $k = 0, 1, \dots, n - 1$.

Selection and Multiplication takes $O(n)$ time. We need to do evaluation and interpolation in sub-$O(n^2)$ time.

Evaluation Divide and Conquer

Suppose we pick plus-minus pairs of x such that we have $\pm x_0, \pm x_1, \ldots, \pm x_{n/2-1}$, squaring the plus-minus pairs gives us the same value. $x_0^2, x_1^2, \ldots, x_{n/2-1}^2$.
Looking at an example,

$$A(x) = 3 + 4x + 6x^2 + 2x^3 + x^4 + 10x^5 = (3 + 6x^2 + x^4) + x(4 + 2x^2 + 10x^4)$$

In the RHS, we have $A_e(x) = 3 + 6x + x^2$ and LHS $A_o(x) = 4 + 2x + 10x^2$. $A_e(\cdot)$ contains the even degree coefficients and $A_o(\cdot)$ contains the odd degree coefficients. In general terms,

$$A(x) = A_e(x^2) + xA_o(x^2)$$

In our example,

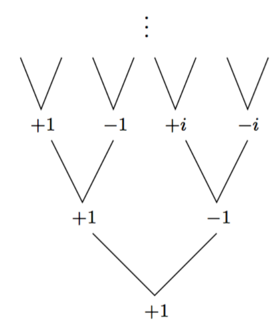
$$A_e(x) = 3 + 6x + x^2$$
$$A_o(x) = 4 + 2x + 10x^2$$

If we use positive-negative pairs x_i,

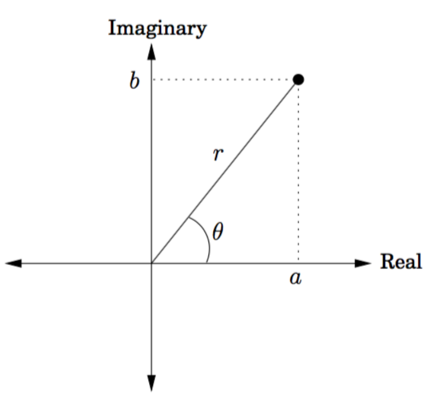
$$A(x_i) = A_e(x^2) + xA_o(x^2)$$
$$A(-x_i) = A_e(x^2) - xA_o(x^2)$$

After the first level, we have to make x_0 and x_1, x_2 and $x_3, \ldots$ positive negative pairs as well. If we can do this until $n = 1$, at each level we make two recursive calls to evaluate a problem that is half the size. Thus we have a recurrence relation $T(n) = 2T(n/2) + O(n)$ and runtime $O(n \log n)$.

Back to finding values of x that we can keep finding pairs such that there will be positive-negative pairs after squaring them. This can be achieved using complex numbers.



Squaring $+1$ and -1 gives us $+1$. Simiarily, squaring $+i$ and $-i$ gives us -1. Now at this level, squaring $+1$ and -1 gives us $+1$.



The complex plane

$z = a + bi$ is plotted at position (a, b).

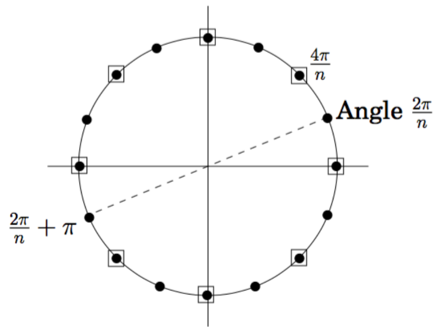
Polar coordinates: rewrite as $z = r(\cos\theta + i\sin\theta) = re^{i\theta}$, denoted (r, θ).

- *length* $r = \sqrt{a^2 + b^2}$.
- *angle* $\theta \in [0, 2\pi)$: $\cos\theta = a/r, \sin\theta = b/r$.
- θ can always be reduced modulo 2π.

Examples:

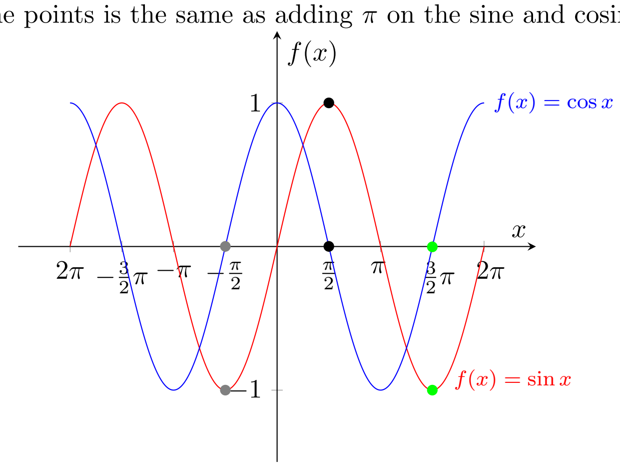
Number	-1	i	$5 + 5i$
Polar coords	$(1, \pi)$	$(1, \pi/2)$	$(5\sqrt{2}, \pi/4)$

If we use nth roots of unity, such that n is a power of two, we can keep squaring pairs at each level. The nth roots of unity are complex numbers $1, \omega, \omega^2, \ldots, \omega^{n-1}$, where $\omega = e^{2\pi i/n}$. When n is even, these roots are plus-minus pairs, $\omega^{n/2+j} = -\omega^j$. Squaring them produces us $(n/2)$nd roots of unity.



These n roots are solutions to the equation $z^n = 1$. Solutions are $z = re^{ie}$ for some multiple of $2\pi/n$. In the unit circle, the numbers are plus-minus paired. $-\cos\theta - i\sin\theta = \cos(\theta+\pi) + i\sin(\theta+\pi)$. The squares will be the $(n/2)$nd roots of unity, which is the immediate left with a box around the point.

Now let us see why adding π will negate the number. Picking a point on the x axis, we can see that negating the points is the same as adding π on the sine and cosine curves.



Below we have the polynomial formulation of the fast Fourier transform. A has polynomial of degree $\leq n-1$.

procedure $\mathrm{FFT}(A, \omega)$
  if $\omega = 1$ **then return** $A(1)$
  Split $A(x)$ into $A_e(x^2) + A_o(x^2)$
  $\mathrm{FFT}(A_e, \omega^2)$
  $\mathrm{FFT}(A_o, \omega^2)$
  for $j = 0, -1$ **do**
    $A(\omega^j) = A_e(\omega^{2j}) + \omega^j A_o(\omega^{2j})$
  return $A(\omega^0), \ldots A(\omega^{n-1})$

Evaluation FFT Example

Let's use our example from earlier.

$$A(x) = 3 + 4x + 6x^2 + 2x^3 + x^4 + 10x^5 = (3 + 6x^2 + x^4) + x(4 + 2x^2 + 10x^4)$$

- **Level 1**
  We see that $A(x)$ has degree 5, so we need the smallest power of two ≥ 6.
  Thus we have $n = 8$ and $\omega = e^{2\pi i/8} = e^{\pi i/4} = \cos(\pi/4) + i\sin(\pi/4)$. Below are the 8 roots of unity in positive negative pairs.

$$\begin{aligned}
\omega^0 &= 1 \\
\omega^4 &= e^{\pi i} = \cos(\pi) + i\sin(\pi) = -1 \\
\omega^1 &= e^{\pi i/4} = \cos(\pi/4) + i\sin(\pi/4) = \frac{1+i}{\sqrt{2}} \\
\omega^5 &= e^{5\pi i/4} = \cos(5\pi/4) + i\sin(5\pi/4) = -\frac{1+i}{\sqrt{2}} \\
\omega^2 &= e^{\pi i/2} = \cos(\pi/2) + i\sin(\pi/2) = i \\
\omega^6 &= e^{3\pi i/2} = \cos(3\pi/2) + i\sin(3\pi/2) = -i \\
\omega^3 &= e^{3\pi i/4} = \cos(3\pi/4) + i\sin(3\pi/4) = -\frac{1-i}{\sqrt{2}} \\
\omega^7 &= e^{7\pi i/4} = \cos(7\pi/4) + i\sin(7\pi/4) = \frac{1-i}{\sqrt{2}}
\end{aligned}$$

  Next we split $A(x)$ into two recursive polynomials.

$$\begin{aligned}
A(x) &= A_e(x^2) + xA_o(x^2) \\
A_e(x) &= B(x) = 3 + 6x + x^2 \\
A_o(x) &= C(x) = 4 + 2x + 10x^2
\end{aligned}$$

  Substituting the roots of unity,

$$\begin{aligned}
A(\omega^0) &= B(1^2) + C(1^2) = B(1) + C(1) \\
A(\omega^4) &= B((-1)^2) - C((-1)^2) = B(1) - C(1) \\
A(\omega^2) &= B(i^2) + iC(i^2) = B(-1) + iC(-1) \\
A(\omega^6) &= B((-i)^2) + iC((-i)^2) = B(-1) - iC(-1) \\
A(\omega^1) &= B\Big((\frac{1+i}{\sqrt{2}})^2\Big) + \frac{1+i}{\sqrt{2}}C\Big((\frac{1+i}{\sqrt{2}}^2)\Big) = B(i) + \frac{1+i}{\sqrt{2}}C(i) \\
A(\omega^5) &= B\Big((-\frac{1+i}{\sqrt{2}})^2\Big) - \frac{1+i}{\sqrt{2}}C\Big((-\frac{1+i}{\sqrt{2}}^2)\Big) = B(i) - \frac{1+i}{\sqrt{2}}C(i) \\
A(\omega^3) &= B\Big((-\frac{1-i}{\sqrt{2}})^2\Big) - \frac{1-i}{\sqrt{2}}C\Big((-\frac{1-i}{\sqrt{2}}^2)\Big) = B(-i) - \frac{1-i}{\sqrt{2}}C(-i) \\
A(\omega^7) &= B\Big((\frac{1-i}{\sqrt{2}})^2\Big) + \frac{1-i}{\sqrt{2}}C\Big((\frac{1-i}{\sqrt{2}}^2)\Big) = B(-i) + \frac{1-i}{\sqrt{2}}C(-i)
\end{aligned}$$

After the recursive call

$$
\begin{aligned}
A(\omega^0) &= B(1) + C(1) = 10 + 16 = 26 \\
A(\omega^4) &= B(1) - C(1) = 10 - 16 = -6 \\
A(\omega^2) &= B(-1) + iC(-1) = -2 + 12i \\
A(\omega^6) &= B(-1) - iC(-1) = -2 - 12i \\
A(\omega^1) &= B(i) + \frac{1+i}{\sqrt{2}}C(i) = 2 + 6i + (\frac{1+i}{\sqrt{2}})(-6+2i) = 2 + 6i - (4+2i)\sqrt{2} \\
A(\omega^5) &= B(i) - \frac{1+i}{\sqrt{2}}C(i) = 2 + 6i - (\frac{1+i}{\sqrt{2}})(-6+2i) = 2 + 6i + (4+2i)\sqrt{2} \\
A(\omega^3) &= B(-i) - \frac{1-i}{\sqrt{2}}C(-i) = 2 - 6i + (\frac{1-i}{\sqrt{2}})(-6-2i) = (2-6i) - (4-2i)\sqrt{2} \\
A(\omega^7) &= B(-i) + \frac{1-i}{\sqrt{2}}C(-i) = 2 - 6i - (\frac{1-i}{\sqrt{2}})(-6-2i) = (2-6i) + (4-2i)\sqrt{2}
\end{aligned}
$$

Thus we have the points that we need.

- **Level 2**
  Both $B(x)$ and $C(x)$ have degree 2 polynomial. Thus we end up with the 4 roots of unity via the recursive call, $\omega = e^{2\pi i/4}$,

$$
\begin{aligned}
\omega^0 &= 1 \\
\omega^2 &= e^{\pi i} = \cos(\pi) + i\sin(\pi) = -1 \\
\omega^1 &= e^{\pi i/2} = \cos(\pi/2) + i\sin(\pi/2) = i \\
\omega^3 &= e^{3\pi i/2} = \cos(3\pi/2) + i\sin(3\pi/2) = -i
\end{aligned}
$$

Again we split both $B(x)$ and $C(x)$ into two halves,

$$
\begin{aligned}
B(x) &= 3 + 6x + x^2 = B_e(x^2) + xB_o(x^2) \\
B_e(x) &= D(x) = 3 + x \\
B_o(x) &= E(x) = 6
\end{aligned}
$$

$$
\begin{aligned}
C(x) &= 4 + 2x + 10x^2 = C_e(x) + xC_o(x^2) \\
C_e(x) &= F(x) = 4 + 10x \\
C_o(x) &= G(x) = 2
\end{aligned}
$$

Substituing the 4 roots of unity,

$$
\begin{aligned}
B(\omega^0) &= D(1^2) + E(1^2) = D(1) + E(1) \\
B(\omega^2) &= D((-1)^2) - E((-1)^2) = D(1) - E(1) \\
B(\omega^1) &= D(i^2) + iE(i^2) = D(-1) + iE(-1) \\
B(\omega^3) &= D((-i)^2) - iE((-i)^2) = D(-1) - iE(-1) \\
C(\omega^0) &= F(1) + G(1) \\
C(\omega^2) &= F(1) - G(1) \\
C(\omega^1) &= F(-1) + iG(-1) \\
C(\omega^3) &= F(-1) - iG(-1)
\end{aligned}
$$

After the recursive call

$$
\begin{aligned}
B(\omega^0) &= D(1) + E(1) = 4 + 6 = 10 \\
B(\omega^2) &= D(1) - E(1) = 4 - 6 = -2 \\
B(\omega^1) &= D(-1) + iE(-1) = 2 + 6i \\
B(\omega^3) &= D(-1) - iE(-1) = 2 - 6i \\
C(\omega^0) &= F(1) + G(1) = 14 + 2 = 16 \\
C(\omega^2) &= F(1) - G(1) = 14 - 2 = 12 \\
C(\omega^1) &= F(-1) + iG(-1) = -6 + 2i \\
C(\omega^3) &= F(-1) - iG(-1) = -6 - 2i
\end{aligned}
$$

- **Level 3**
  Now we are left with 2 roots of unity for functions $D(x), E(x), F(x), G(x)$. At this point, we can just do arithmetic. However, I will show how the algorithm continues until the next level, which is the base case.
  Again we split the 4 functions into halves,

$$
\begin{aligned}
D(x) &= 3 + x = D_e(x^2) + xD_o(x^2) \\
D_e(x) &= 3 \\
D_o(x) &= 1 \\
E(x) &= 6 = E_e(x^2) + xE_o(x^2) \\
E_e(x) &= 6 \\
E_o(x) &= 0 \\
F(x) &= 4 + 10x = F_e(x^2) + xF_o(x^2) \\
F_e(x) &= 4 \\
F_o(x) &= 10 \\
G(x) &= 2 = G_e(x^2) + xG_o(x^2) \\
G_e(x) &= 2 \\
G_o(x) &= 0
\end{aligned}
$$

We have 2 roots of unity,

$$
\begin{aligned}
D(\omega^0) &= D_e(1^2) + D_o(1^2) = D_e(1) + D_o(1) \\
D(\omega^1) &= D_e((-1)^2) + D_o((-1)^2) = D_e(1) - D_o(1) \\
E(\omega^0) &= E_e(1) + E_o(1) \\
E(\omega^1) &= E_e(1) - E_o(1) \\
F(\omega^0) &= F_e(1) + F_o(1) \\
F(\omega^1) &= F_e(1) - F_o(1) \\
G(\omega^0) &= G_e(1) + G_o(1) \\
G(\omega^1) &= G_e(1) - G_o(1)
\end{aligned}
$$

After the recursive calls

$$D(\omega^0) = D(1) = 3 + 1 = 4$$
$$D(\omega^1) = D(-1) = 3 - 1 = 2$$
$$E(\omega^0) = E(1) = 3 + 0 = 6$$
$$E(\omega^1) = E(-1) = 3 - 0 = 6$$
$$F(\omega^0) = F(1) = 4 + 10 = 14$$
$$F(\omega^1) = F(-1) = 4 - 10 = -6$$
$$G(\omega^0) = G(1) = 2 + 0 = 2$$
$$G(\omega^1) = G(-1) = 2 - 0 = 2$$

- **Level 4 - Base Case**
  At $\omega = 1$, we just return our functions with 1 passed in. Now we can propagate upwards.

$$D_e(1) = 3, D_o(1) = 1$$
$$E_e(1) = 6, E_o(1) = 0$$
$$F_e(1) = 4, F_o(1) = 10$$
$$G_e(1) = 2, G_o(1) = 0$$

Interpolation

After obtaining values, we need to get it back to coefficients. Let's take a look at the following matrix.

$$\begin{bmatrix} A(x_0) \\ A(x_1) \\ \vdots \\ A(x_{n-1}) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \omega & \omega^2 & \dots & \omega^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega^{n-1} & \omega^{2(n-1)} & \dots & \omega^{(n-1)(n-1)} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{n-1} \end{bmatrix}$$

Let's call the middle matrix $M_n(\omega)$. In this special ordering, we have a Vandermonde matrix. If $\omega^0, \omega^1, \dots, \omega^{n-1}$ are distinct, $M_n(\omega)$ is invertible. Thus we can obtain the coefficients using

$$(M_n(\omega))^{-1} \begin{bmatrix} A(x_0) \\ A(x_1) \\ \vdots \\ A(x_{n-1}) \end{bmatrix} = \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{n-1} \end{bmatrix}$$

We need to find $(M_n(\omega))^{-1}$ such that $M_n(\omega)(M_n(\omega))^{-1} = I_n$.

Lets try $M_n(\omega)M_n(\omega^{-1})$.

$$Z = \begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \omega & \omega^2 & \dots & \omega^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega^{n-1} & \omega^{2(n-1)} & \dots & \omega^{(n-1)(n-1)} \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \omega^{-1} & \omega^{-2} & \dots & \omega^{-(n-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega^{-(n-1)} & \omega^{-2(n-1)} & \dots & \omega^{-(n-1)(n-1)} \end{bmatrix}$$

For (row, column) (j, k), we have

$$\begin{aligned} Z_{(j,k)} &= \sum_{m=0}^{n-1} \omega^{m(j-1)} \omega^{-m(k-1)} \\ &= \sum_{m=0}^{n-1} \omega^{m(j-k)} \\ &= \sum_{m=1}^{n} \omega^{(m-1)(j-k)} \end{aligned}$$

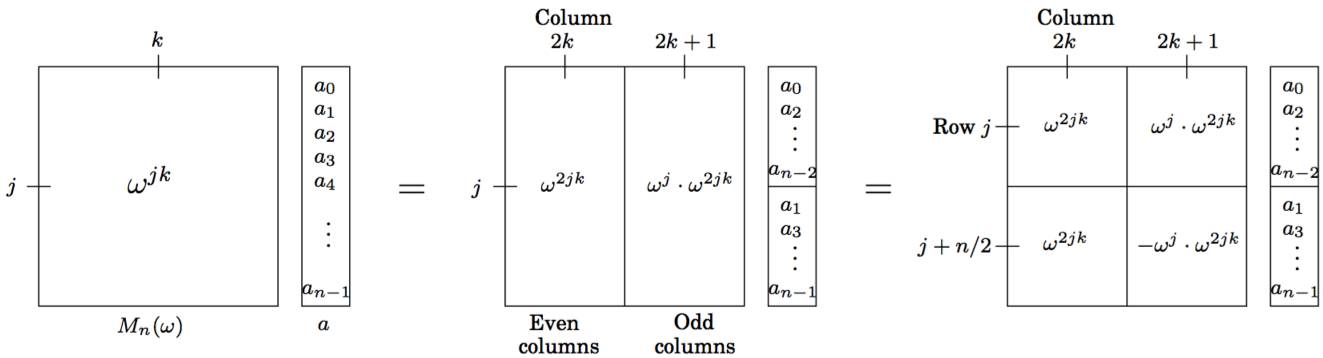
This becomes a geometric series with $r = \omega^{j-k}$. When $j = k$, $Z = n$, which is the term for the entries on the diagonal of the matrix.
When $j \neq k$

$$\begin{aligned} \sum_{m=1}^{n} \omega^{(m-1)(j-k)} &= \frac{1 - (\omega^{(j-k)})^n}{1 - \omega^{(j-k)}} \\ &= \frac{1 - \omega^{n(j-k)}}{1 - \omega^{(j-k)}} \\ \omega &= e^{2\pi i/n} \\ Z_{(j,k)} &= \frac{1 - e^{2(j-k)\pi i}}{1 - e^{2(j-k)\pi i/n}} \\ e^{2(j-k)\pi i} &= \cos(2(j-k)\pi) + i \sin(2(j-k)\pi) \\ &= 1 + i0 \\ &= 1 \\ \frac{1 - e^{2(j-k)\pi i}}{1 - e^{2(j-k)\pi i/n}} &= \frac{0}{1 - e^{2(j-k)\pi i/n}} \\ \therefore Z_{(j,k)} &= 0, j \neq k \end{aligned}$$

Thus

$$
\begin{aligned}
M_n(\omega)M_n(\omega^{-1}) &= nI_n \\
M_n(\omega)\frac{1}{n}M_n(\omega^{-1}) &= I_n \\
\therefore M_n(\omega)^{-1} &= \frac{1}{n}M_n(\omega^{-1})
\end{aligned}
$$

Matrix Form FFT



$$M_n(\omega) = \begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \omega & \omega^2 & \dots & \omega^{(n-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega^{(n-1)} & \omega^{2(n-1)} & \dots & \omega^{(n-1)(n-1)} \end{bmatrix} = \begin{bmatrix} \omega^{jk} \end{bmatrix}$$

First, let's split the matrix where the even index columns $2k$ are on the left side and the odd index columns $(2k+1)$ are on the right side, $0 \leq k \leq n/2$.

$$\begin{bmatrix} 1 & 1 & 1 & \dots & 1 & \\ 1 & \omega^2 & \dots & \omega^1 & \omega^3 & \dots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & \omega^{2(n-1)} & \dots & \omega^{(n-1)} & \omega^{3(n-1)} & \dots \end{bmatrix} = \begin{bmatrix} \omega^{-2jk} & \omega^{-j-2jk} \end{bmatrix} = \begin{bmatrix} \omega^{-2jk} & \omega^{-j} \cdot \omega^{-2jk} \end{bmatrix}$$

Since the column range k has decreased by a half, each element ω^{jk} increases to ω^{2jk}. For each k, the difference between the even column and the odd column is by a multiplicative factor of ω^j. Thus we multiply the even column elements by ω^j to obtain the odd column elements.

Now, lets split the matrix up and bottom. Row index is now $0 \leq j \leq n/2$. Upper portion row indices are j. Lower portion row indices are $j + n/2$.

By decreasing the domain of j by a half, the difference between the lower right half and the upper right half is $j = n/2$. Thus the difference is a multiplicative factor of $\omega^{n/2}$, which is -1 as shown below.

$$\begin{aligned} \omega^n &= (e^{2\pi i/n})^n \\ &= e^{2\pi i} \\ &= \cos 2\pi + i \sin 2\pi \\ &= 1 \\ \omega^{kn} &= e^{2k\pi i} \\ &= \cos 2k\pi + i \sin 2k\pi \\ &= 1 \\ \omega^{n/2} &= (e^{2\pi i/n})^{n/2} \\ &= e^{\pi i} \\ &= \cos \pi + i \sin \pi \\ &= -1 \\ \omega^{kn/2} &= e^{k\pi i} \\ &= \cos k\pi + i \sin k\pi \\ &= -1 \end{aligned}$$

Take $j = 1$ for example, we set the LHS as $-1\cdot$ upper right elements, and set RHS as lower right elements.

$$\begin{aligned}
-1(\omega \cdot \omega^{2k}) &= \omega^{(1+n/2)} \cdot \omega^{2(1+n/2)k} \\
-1(\omega \cdot \omega^{2k}) &= \omega^{1+n/2} \cdot \omega^{(2+n)k} \\
-1(\omega \cdot \omega^{2k}) &= \omega \cdot \omega^{n/2} \cdot \omega^{2k} \cdot \omega^{kn} \\
-1(\omega \cdot \omega^{2k}) &= -1 \cdot \omega \cdot \omega^{2k}
\end{aligned}$$

Thus to obtain the lower right half elements, we multiply the upper right half elements by $\omega^{n/2} = -1$.
Similarly, we can see that the multiplicative difference of the upper left elements and the lower left elements is only 1. Using the $j = 1$ example.

$$\begin{aligned}
\omega^{2j_u k} &= \omega^{2k} \\
\omega^{2j_l k} &= \omega^{2(1+n/2)k} \\
&= \omega^{(n+2)k} \\
&= \omega^{kn} \cdot \omega^{2k} \\
&= 1 \cdot \omega^{2k} \\
&= \omega^{2j_u k}
\end{aligned}$$

With the multiplicative factor between the upper left and lower left being 1, we can leave as is.

$$\begin{bmatrix}
1 & 1 & 1 & \dots & 1 & \\
1 & \omega^2 & \dots & \omega & \omega^3 & \dots \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
1 & \omega^{2(n-1)} & \dots & \omega^{(n-1)} & \omega^{3(n-1)} & \dots
\end{bmatrix} = \begin{bmatrix}
\omega^{2jk} & \omega^j \cdot \omega^{2jk} \\
\omega^{2jk} & -\omega^j \cdot \omega^{2jk}
\end{bmatrix}$$

With all four corners sharing elements ω^{2jk}, such that $0 \leq j \leq n/2$ and $0 \leq k \leq n/2$, we have a $n/2$ x $n/2$ matrix.

$$\begin{bmatrix}
1 & 1 & 1 & \dots & 1 \\
1 & \omega^2 & \omega^4 & \dots & \omega^{(n-2)} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & \omega^n & \omega^{2n} & \dots & \omega^{(n-2)(n-2)}
\end{bmatrix} = M_{n/2}(\omega)$$

$$\therefore M_n(\omega) = \begin{bmatrix}
M_{n/2}(\omega) & \omega^j M_{n/2}(\omega) \\
M_{n/2}(\omega) & -\omega^j M_{n/2}(\omega)
\end{bmatrix}$$

[1]

[1] Diagrams from Course Textbook, Algorithms by Dasgupta, Papadimitriou, and Vazirani